On the Dynamics of a Simple Conservative System

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SYNOPSIS:

This report treats the dynamical system generated by the second-order non-linear differential equation:

$$\frac{d^2x}{dt^2}(t) = \begin{cases} 0, & \text{if } x(t) \leq 2 \text{ or } x(t) > 2; \\ x(t)+2, & \text{if } -2 < x(t) \leq -1; \\ -x(t)^3, & \text{if } -1 < x(t) \leq 1; \\ x(t)-2, & \text{if } 1 < x(t) \leq 2. \end{cases}$$

We show that this system admits an energy function; existence and uniqueness for general nth-order systems is treated and it is shown that the above system has unique, globally defined solutions. Explicit solutions are obtained for the cases $x(t) \leq -1$ and $x(t) > 1$ and using these together with the energy function, we classify the fixed points. Finally, we show that solutions corresponding to negative energies are periodic.

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Preface

This report is the result of the project work done in the third semester of the B.Sc. study at the Department of Mathematical Sciences, Aalborg University under the general theme *dynamical systems*. Our work is based on a project proposal given by Horia Cornean.

Citations will appear as [author, year, page/theorem].

Of special notation, vectors and mappings in Euclidean space are written with bold-faced characters. Assignments will be made using the symbol ":=", e.g. $x := 2$. Further use of notation will be stated in the report.

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Resume

In this report, we study a simple dynamical system arising from a variant of Newton's equation of motion in one dimension, a second order differential equation.

The introduction is a discussion of the basic characteristics of the system. We characterize it as conservative, and prove that there exists an energy function which is constant for any solution to the system. Using this, function, we visualize the system dynamics and give a brief graphical analysis.

Chapter 2 contains a general discussion of representation of systems of differential equations of arbitrary order. It is proved that the problem of determining a solution to an nth order system can be reduced to solving a first-order system. The initial value problem (IVP) for a first order system is then defined, and it is shown that under certain continuity assumptions the IVP can be represented as an integral equation.

In order to study existence of solutions to this integral equation, we treat existence and uniqueness for fixed points of general operators acting on metric spaces and introduce the concept of normed spaces and Banach spaces as a way to characterize spaces of functions. We study convergence criteria for sequences in such spaces.

These general results are used to prove that if the function defining the system of DEs is sufficiently "smooth", a unique, local solution to an IVP always exists. We show that this can be extended to a unique solution defined on the largest interval of times possible. Sufficient criteria for the solution to be globally defined are presented, and it is shown that our system fulfills these.

In chapter 5, we introduce the concept of the operator exponential in a general setting and show how this can be used in the case of operators on $\mathbb{R}^N$ to form the matrix exponential. Using the matrix exponential, an explicit formula for the solution of linear, inhomogeneous, constant-coefficient systems is given and applied to our own system in certain linear regions.

Chapter 6 concerns the general dynamics of our system. We define stability of equilibrium points for autonomous systems and give a simple proof that the origin is a stable equilibrium point using the properties of the energy function. The remaining equilibrium points are shown to be unstable; the proof of this partly relies on the fact that we are able to determine explicit solutions in this region.

Using the existence of an energy function, we conclude that solutions which correspond to negative values of this function are periodic and we devise an explicit formula for calculating the period of these.

Finally, possible generalizations are briefly discussed, and we argument that the dynamics of our system depends only on a specific number of properties, so that our investigation will apply equally well to systems with similar properties. A simple example of a different system with similar dynamics is described and a graphical comparison is made.

Contents

CHAPTER 1

Introduction

One possible way to describe one dimensional motion of a particle is via Newton's equations of motion. Expressed as a differential equation and in the general form, it is

$$\frac{d^2x}{dt^2}(t) = \frac{1}{m}F(t, x(t), \frac{dx}{dt}(t)), \quad t \geq 0, \tag{1.1}$$

where m is the mass of the particle and F is the *force function*. In this report we will investigate this equation for the case where the force is given by $x(t) \mapsto F(x(t))$ with $F : \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) := \begin{cases} 0 & \text{if } x \leq -2 \text{ or } x > 2, \\ x + 2 & \text{if } -2 < x \leq -1, \\ -x^3 & \text{if } -1 < x \leq 1, \\ x - 2 & \text{if } 1 < x \leq 2. \end{cases} \tag{1.2}$$

This yields a simple system in which the right-hand side of (1.1) does not (explicitly) depend on the time t but only on the position. Without loss of generality, we may assume that $m = 1$ (this is simply a rescaling of the right-hand side of (1.1)), and we arrive at the following equation

$$\frac{d^2x}{dt^2}(t) = F(x(t)), \quad t \geq 0. \tag{1.3}$$

The purpose of this report is to determine the *qualitative behaviour* — the dynamics — of the trajectories of the particle when it is influenced by F.

The following section provides a basic analysis of a few simple but very important properties of (1.3); properties which give at least an idea of the behaviour of the trajectories.

1.1 Preliminary Analysis

1.1.1 Analysis of the Force Function

In order to apply the general methods for continuous dynamical systems determined by differential equations, we must ensure that the force function F as in (1.2) is indeed a continuous function. Observe that this implies that $x(t) \mapsto F(x(t))$ is also continuous, since $x(t)$ by assumption should be twice differentiable and therefore continuous, [Apostol, 1974, Theorem 5.3]. While this is not difficult to verify, we can actually impose further structure on F. In fact, there exists a differentiable function $V : \mathbb{R} \to \mathbb{R}$ satisfying

$$-\frac{dV}{dt}(x(t)) = F(x(t)), \quad t \geq 0. \tag{1.4}$$

We claim that this C^1-function is given by

$$V(x) := \begin{cases} 0 & \text{if } x \leq -2 \text{ or } x > 2, \\ -2x - x^2/2 - 2 & \text{if } -2 < x \leq -1, \\ x^4/4 - 3/4 & \text{if } -1 < x \leq 1, \\ 2x - x^2/2 - 2 & \text{if } 1 < x \leq 2. \end{cases} \tag{1.5}$$

In physical terms, this is the *potential energy*, cf. [Arnold, 1980, p. 15]. By standard differentiation of polynomials, (1.4) holds in between the region jumps of V, i.e. the points $\{-2, -1, 1, 2\}$. Therefore, it is sufficient to verify the differentiability claim for these points.

By the remark following [Apostol, 1974, Definition 5.6], V is differentiable at a point x iff the left- and right-hand derivatives exist and are equal at x. We verify this for the point $x = 1$ by employing the expressions of $V'(x) = -F(x)$ for respectively the left- and right-hand side of the point $x = 1$

$$\lim_{x \uparrow 1} x^3 = 1 = \lim_{x \downarrow 1} (-x + 2).$$

Hence V is differentiable at $x = 1$ and also $-V'(1) = F(1)$; the procedure for the remaining points is similar.

Also, F must be a continuous function. This follows from the fact that F clearly is continuous between the region jumps, since polynomials determine continuous functions (they arise from elementary operations on the functions $x \mapsto x$ and $x \mapsto 1$). Regarding the region jumps, $x = \pm 2$, $x = \pm 1$, the above results concerning left- and right-hand derivatives show that $V'(x) = -F(x)$ is continuous at these points, which implies continuity of F.

We conclude that it is possible to rewrite (1.3) as

$$\frac{d^2x}{dt^2}(t) = -\frac{dV}{dt}(x(t)), \quad t \geq 0. \tag{1.6}$$

1.1.2 Basic Dynamical Properties

Notation. For the sake of notational simplicity, we will mostly use the notation x for the function $x(t)$. It will be clear from the context when we are talking about a function dependent on t.

Conservation Property

Suppose there exists some function $q : \mathbb{R} \to \mathbb{R}$ which satisfies (1.6) (this is of course a necessary condition for q to solve (1.6), a precise definition is given in section 2.1). Multiplying both sides of the equation with $\dot{q}$ and applying the chain rule of differentiation [Apostol, 1974, Theorem 5.5], we obtain

$$\dot{q}\ddot{q} = -V'(q)\dot{q} \Leftrightarrow \frac{d}{dt}\left(\frac{\dot{q}^2}{2} + V(q)\right) = 0, \quad \forall\, t \geq 0.$$

Since only a function, which is identically constant has zero derivative, we conclude that the function $\tilde{E} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\tilde{E}(q,\dot{q}) = \frac{\dot{q}^2}{2} + V(q), \tag{1.7}$$

is constant whenever the curve q satisfies (1.6), $\tilde{E}$ is *conserved.* Observe, that we can also view $\tilde{E}$ as the composite function $t \mapsto (q,\dot{q}) \mapsto \tilde{E}$. We will denote this function by E.

The function $\tilde{E}$ is called the *energy function* or the *total energy* due to its physical interpretation, cf. section 1.1.3.

Graphical Representation of Trajectories

The conservation property enables us to formulate another necessary condition for a function q to solve the differential equation — if q satisfies (1.6), then the curve $\tilde{q} = (q,\dot{q})$ lies on one of the level curves of $\tilde{E}(q,\dot{q})$, that is, the curves defined by

$$\tilde{E}(q,\dot{q}) = \text{constant}.$$

Let us assume that $\dot{q}(0) = a > 0$ and $q(0) = 0$. This yields the following expression of the energy function

$$\tilde{E}(q,\dot{q}) = E(0) = \frac{a^2}{2} - \frac{3}{4}. \tag{1.8}$$

Equation (1.8) is useful for visualising how the trajectories in the *configuration plane* (the position-velocity plane) of the system (1.1) look like, for different values of a. In figure 1.1, a plot of the general level curves of $\tilde{E}$ is given together with a few curves that satisfy the assumption $t \geq 0$, respectively $\tilde{E}(q,\dot{q}) \leq 0$ and $\tilde{E}(q,\dot{q}) > 0$.

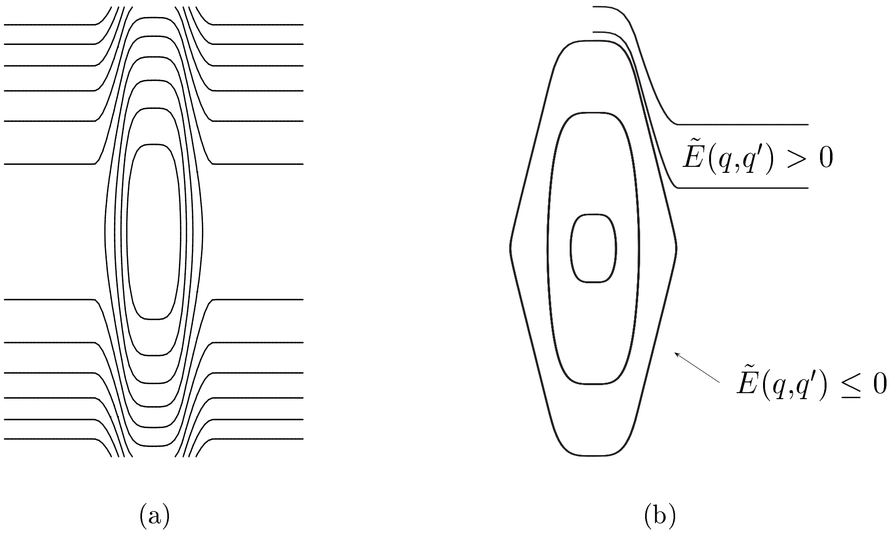

Figure 1.1: *Figure (a) displays the general level curves of $\tilde{E}$ in the configuration plane. Figure (b) displays a few plots of the special case, where we assume $t \geq 0$, $q(0) = 0$. Notice, that for $\tilde{E}(q,\dot{q}) > 0$ the level curves are no longer closed (the figures are scaled differently).*

What can be drawn from this particular plot? Apparently, the curves for which $\tilde{E}(q,\dot{q}) \leq 0$ are closed curves. This will correspond to periodic motion. Whenever $q(t) = 0$, the velocity is maximal and correspondingly, if $\dot{q}(t) = 0$, the position is maximal. Letting $\tilde{E}(q,\dot{q}) > 0$, the level curves seem to follow the path of the closed curves a while but eventually escape and continue with constant velocity.

These phenomenons have an intuitive explanation, if we apply a physical interpretation of the problem.

1.1.3 Briefly on the Physical Interpretation

A physical interpretation tells us that the conservation property of our system is the usual conservation of energy in physical systems. The energy function, cf. (1.7), is in fact the *total mechanical energy* of the particle, that is, the sum of the kinetic energy and the potential energy, cf. [Arnold, 1980, p. 16].

This is in agreement with our result of the previous section, which tells that every curve which satisfies the differential equation also makes $\tilde{E}(q,\dot{q})$ constant for all $t \geq 0$, i.e. the mechanical energy of the particle is a *constant of motion*. A system which has this property is called *conservative*, cf. [Arnold, 1980, p. 22].

The existence of an energy function allows us to view the problem as a *particle trapped in a potential well* given by the function V in (1.5). The potential well is depicted in figure 1.2. The particle will only be able to escape (to infinity), if its initial velocity is sufficiently large, making the mechanical energy greater than zero. This is the physical explanation of the phenomenon occurring in figure 1.1 where, indeed, the energy curves (and hence solution curves) are no longer closed if the mechanical energy is greater than zero.

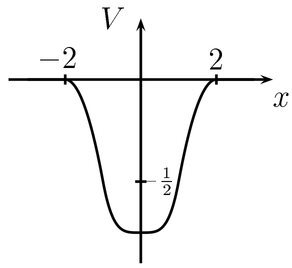

Figure 1.2: *The potential well in which the particle is trapped.*

Also, if the mechanical energy is less than zero, the particle should intuitively exhibit periodic motion; since when the potential energy increases, the kinetic energy must decrease correspondingly eventually making the particle stop and move in the opposite direction, yielding periodic motion.

1.2 Further Analysis

In this introductory chapter we have presented a combined visual/physical interpretation of the dynamics of the Newtonian system in consideration.

This introduction gave some insight into the dynamics of the system. The conservation property of the force function $F(x)$ made it possible to obtain parametrized solution curves in the position-velocity plane; and using different initial data we obtained a simple graphical description of the dynamics. The phenomenons observed in this graphical description seemed meaningful and convincing when applying a physical interpretation.

Based on this analysis, we will now investigate the following questions in a more precise setting, partly for the general case, partly for our own system:

(i) Does the system (1.3) determine trajectories that exist for all time — if so, are they uniquely determined?

(ii) What is the significance of the origin and the points $x = \pm p$, $2 \leq p$; the points where the force function F is zero?

(iii) Do periodic trajectories actually exist; if so, under which conditions?

CHAPTER 2

Basics of Differential Equations

We will now give a precise formulation of the notion of a *system of ordinary differential equations* (system of ordinary DEs), concentrating on giving a variety of useful representations that eventually will allow us to deal with a single type of equation, (a so-called *integral equation*).

A system of differential equations is, loosely speaking, a functional relation between a curve and one or more of its derivatives.

Definition 2.1 (System of differential equations). *Let I be an open interval and suppose $\boldsymbol{x} : I \to \mathbb{R}^N$ is n times differentiable. Assume that*

$$U \subseteq \mathbb{R} \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N = \mathbb{R} \times \left(\mathbb{R}^N\right)^{(n+1)},$$

is open and let $\boldsymbol{G} : U \to \mathbb{R}^N$ be a mapping.

An nth-order system of ordinary differential equations is a functional equation

$$\boldsymbol{G}(t,\boldsymbol{x},\boldsymbol{x}',\ldots,\boldsymbol{x}^{(n)}) = \boldsymbol{0}.$$

The above is the most general form of an ordinary differential equation; the assumption that I be open is necessary since a function is left- and right-differentiable (and thus differentiable, according to the remark following [Apostol, 1974, Definition 5.6]) on open intervals only. If the function $\boldsymbol{G}$ in definition 2.1 does not depend explicitly on t, the system is called *autonomous*.

Notation. We will use the general notion of a *curve* meaning a function defined on an open interval and with values in $\mathbb{R}^N$.

Remark. The general notion of a differential equation involves *both* ordinary and partial differential equations. The latter type involves unknown functions of several variables and will not be considered in this report. By "differential equation" we always mean an *ordinary differential equation.*

2.1 Solutions and Representation

Assuming that it is possible to isolate the derivative of highest order in the general ordinary differential equation, we arrive at useful representation; the so-called *normal form.*

Definition 2.2 (Normal form). *Assume that for an nth-order system of differential equations, it is possible to express* $\boldsymbol{x}^{(n)}$ *as a function* $\boldsymbol{F}$ *of* $t,\boldsymbol{x},\boldsymbol{x}',\ldots,\boldsymbol{x}^{(n-1)}$, $\boldsymbol{F}:\mathcal{O}\to\mathbb{R}^N$, *where* $\mathcal{O}\subseteq\mathbb{R}\times\left(\mathbb{R}^N\right)^n$ *is open. Then the system of differential equations given by*

$$\boldsymbol{x}^{(n)}=\boldsymbol{F}(t,\boldsymbol{x},\boldsymbol{x}',\ldots,\boldsymbol{x}^{(n-1)}),$$

is said to be an nth-order system of DEs on normal form.

From this point on we will only deal with systems of DEs on normal form — and when we refer to a system of differential equations, it is *always* assumed to be on normal form. It can readily be seen that our system, (1.3), is already on normal form.

We are now in position to give a precise definition, of what is meant by a solution to a system of DEs.

Definition 2.3 (Solution to a system of DEs). *Consider the nth-order system*

$$\boldsymbol{x}^{(n)}=\boldsymbol{F}(t,\boldsymbol{x},\boldsymbol{x}',\ldots,\boldsymbol{x}^{(n-1)}), \tag{2.1}$$

where $\boldsymbol{F}:\mathcal{O}\to\mathbb{R}^N$.

A curve $\boldsymbol{\alpha}:I\to\mathbb{R}^N$ *is said to be a solution to* (2.1) *if it satisfies*

(i) $\boldsymbol{\alpha}$ *is n times differentiable on I;*

(ii) $(t,\boldsymbol{\alpha},\boldsymbol{\alpha}',\ldots,\boldsymbol{\alpha}^{(n-1)})\in\mathcal{O}$ *for all* $t\in I$*;*

(iii) $\boldsymbol{\alpha}^{(n)}=\boldsymbol{F}(t,\boldsymbol{\alpha},\boldsymbol{\alpha}',\ldots,\boldsymbol{\alpha}^{(n-1)})$ *for all* $t\in I$.

Although we have restricted ourselves to systems on normal form, it is clear that a more general representation of systems of differential equations of arbitrary order is desirable, if we are to prove anything in general.

It turns out that every nth-order system (on normal form) can be rewritten to a system of first-order equations — and that this system and the original system are equivalent, in the sense that once a solution to one of the systems has been found, one can immediately find a solution to the other.

We first define what is meant by a corresponding first-order system.

Definition 2.4 (Corresponding first-order system). *Let $\boldsymbol{F} : \mathcal{O} \to \mathbb{R}^N$ and consider the nth-order system*

$$\boldsymbol{x}^{(n)} = \boldsymbol{F}(t,\boldsymbol{x},\boldsymbol{x}',\ldots,\boldsymbol{x}^{(n-1)}).$$

with $\boldsymbol{x} : I \to \mathbb{R}^N$. The corresponding first-order system is defined as

$$\boldsymbol{y}' = \boldsymbol{X}(t,\boldsymbol{y}),$$

where $\boldsymbol{y} : I \to \left(\mathbb{R}^N\right)^n$ with $\boldsymbol{y} := (\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_n)$ and $\boldsymbol{X} : \mathcal{O} \to \left(\mathbb{R}^N\right)^n$ is given by

$$\boldsymbol{X}(t,\boldsymbol{y}) := (\boldsymbol{y}_2,\boldsymbol{y}_3,\ldots,\boldsymbol{y}_n,\boldsymbol{F}(t,\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_n)).$$

Since the process of obtaining the corresponding first-order system is merely a matter of introducing sufficiently many new variables, it is not surprising that the new and the original system are in some sense equivalent.

Proposition 2.1. *Consider the nth-order system*

$$\boldsymbol{x}^{(n)} = \boldsymbol{F}(t,\boldsymbol{x},\boldsymbol{x}',\ldots,\boldsymbol{x}^{(n-1)}), \tag{2.2}$$

with $\boldsymbol{F} : \mathcal{O} \to \mathbb{R}^N$. A curve $\boldsymbol{\alpha} : I \to \mathbb{R}^N$ is a solution to (2.2) *if, and only if*

$$\tilde{\boldsymbol{\alpha}} := (\boldsymbol{\alpha},\boldsymbol{\alpha}',\ldots,\boldsymbol{\alpha}^{(n-1)})$$

is a solution to the corresponding first-order system.

Proof. First, suppose $\boldsymbol{\alpha}$ solves (2.2). By assumption, $\boldsymbol{\alpha}$ is n times differentiable so $\tilde{\boldsymbol{\alpha}}' = (\boldsymbol{\alpha}',\boldsymbol{\alpha}'',\ldots,\boldsymbol{\alpha}^{(n)})$ exists. We also have

$$(t,\boldsymbol{\alpha},\boldsymbol{\alpha}',\ldots,\boldsymbol{\alpha}^{(n-1)}) = (t,\tilde{\boldsymbol{\alpha}}) \in \mathcal{O} \text{ for all } t \in I.$$

Writing $\boldsymbol{y}' = \boldsymbol{X}(t,\boldsymbol{y})$ in component form

$$\begin{aligned}
\boldsymbol{y}_1' &= \boldsymbol{y}_2 \\
\boldsymbol{y}_2' &= \boldsymbol{y}_3 \\
&\vdots \\
\boldsymbol{y}_n' &= \boldsymbol{F}(t, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n).
\end{aligned}$$

Set $\boldsymbol{y}_1 = \boldsymbol{\alpha}, \boldsymbol{y}_2 = \boldsymbol{\alpha}', \ldots, \boldsymbol{y}_n = \boldsymbol{\alpha}^{(n-1)}$. Now it is easily seen that all the equations are satisfied, in particular

$$\boldsymbol{\alpha}^{(n)} = \boldsymbol{F}(t, \boldsymbol{\alpha}, \boldsymbol{\alpha}', \ldots, \boldsymbol{\alpha}^{(n-1)}),$$

which is satisfied by assumption. Therefore $\tilde{\boldsymbol{\alpha}}$ solves $\boldsymbol{y}' = \boldsymbol{X}(t,\boldsymbol{y})$.

To prove the converse, assume $(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2, \ldots ,\boldsymbol{\beta}_n)$ solves the corresponding first-order system, $\boldsymbol{y}' = \boldsymbol{X}(t,\boldsymbol{y})$. This implies

$$\begin{aligned}
\boldsymbol{\beta}_1' &= \boldsymbol{\beta}_2 \\
\boldsymbol{\beta}_2' &= \boldsymbol{\beta}_3 \\
&\vdots \\
\boldsymbol{\beta}_n' &= \boldsymbol{F}(t, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_n).
\end{aligned}$$

Now $(t, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_n) \in \mathcal{O}$ for all $t \in I$, so $(t, \boldsymbol{\beta}_1, \boldsymbol{\beta}_1', \ldots, \boldsymbol{\beta}_1^{(n-1)}) \in \mathcal{O}$ for all $t \in I$.

Finally since $\boldsymbol{\beta}_k = \boldsymbol{\beta}_1^{(k-1)}$ we have that

$$\boldsymbol{\beta}_n' = \boldsymbol{\beta}_1^{(n)} = \boldsymbol{F}(t, \boldsymbol{\beta}_1, \boldsymbol{\beta}'_1, \ldots, \boldsymbol{\beta}_1^{(n-1)}),$$

so $\boldsymbol{\beta}_1$ solves (2.2). □

Notation. A mapping $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$, $\mathcal{O} \subseteq \mathbb{R}^N$ is called an autonomous vector field — if $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^N$, the vector field is called time-dependent (or non-autonomous). Correspondingly, given the first order system

$$\boldsymbol{x}' = \boldsymbol{X}(t,\boldsymbol{x}),$$

we will speak of the non-autonomous system of DEs *generated* by the vector field $\boldsymbol{X}$. Unless otherwise stated, we shall assume that all vector fields are time-dependent.

The above method easily applies to our own (second-order) system, (1.3), with F defined as in (1.2). We introduce one new variable, say v and let $\boldsymbol{z} : \mathbb{R}_0^+ \to \mathbb{R}^2$ be given by $\boldsymbol{z} := (x,v)$. This gives the following first-order system

$$\left.\begin{aligned} x' &= v \\ v' &= F(x) \end{aligned}\right\} \tag{2.3}$$

or equivalently

$$\boldsymbol{z}' = \boldsymbol{X}(\boldsymbol{z}) = (v,F(x)).$$

To summarize, if $\alpha : I \to \mathbb{R}$ solves (1.3), then by proposition 2.1 we have

$$\tilde{\boldsymbol{\alpha}} = (\alpha, \alpha'),$$

is a solution to (2.3).

2.1.1 Linear Systems of DEs

A first-order system of differential equations (on normal form)

$$\boldsymbol{x}' = \boldsymbol{X}(t,\boldsymbol{x}),$$

with $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$, is called *linear* if the vector field $\boldsymbol{X}$ is a linear mapping, i.e. if for $(t,\boldsymbol{\alpha}),(t,\boldsymbol{\beta}) \in \mathcal{O}$ we have

$$\boldsymbol{X}(t,c_1\boldsymbol{\alpha} + c_2\boldsymbol{\beta}) = c_1\boldsymbol{X}(t,\boldsymbol{\alpha}) + c_2\boldsymbol{X}(t,\boldsymbol{\beta}), \quad c_1, c_2 \in \mathbb{R}.$$

By the discussion in [Axler, 1997, p. 49], any linear mapping in Euclidean space can be represented by a matrix, so the general linear first-order system can be represented as

$$\boldsymbol{x}' = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{b}(t),$$

where $\boldsymbol{A}(t)$ is an $N \times N$ matrix function (w.r.t. the standard basis of $\mathbb{R}^N$). Of course, for the autonomous case $\boldsymbol{b}(t) \equiv 0$. The system is said to have constant coefficients if $\boldsymbol{A}(t) \equiv \boldsymbol{A}$ for some $N \times N$ matrix $\boldsymbol{A}$.

Our system (of first-order equations), (2.3), is itself clearly linear in the regions $-\infty < x \leq -1$ and $1 < x < \infty$; in fact our system (1.3) reduces to well-known constant coefficients second-order equations, cf. [Jensen, 2000, Sætning 5.3]. In section 5.3, we will determine explicit solutions in these regions using methods of linear algebra which apply to *any* linear system.

Note that if the system is *not* linear (which is the case for (2.3) in the remaining areas), there exists no general methods for obtaining closed-form solutions.

2.2 The Initial Value Problem

Since differentiation will allow the solutions to differ by a constant vector, one would expect a system of first-order DEs to possess infinitely many solutions. However, if we couple the system of DEs with a so-called *initial condition* which a solution must fulfill, we will (with additional hypotheses on the vector field) be able to find a unique solution (see chapter 4).

Definition 2.5 (Initial value problem). *Let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ be a vector field. An initial value problem (IVP) is a first-order system of DEs together with an initial condition, i.e.*

$$\left.\begin{aligned} \boldsymbol{x}' &= \boldsymbol{X}(t,\boldsymbol{x}) \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0, \end{aligned}\right\} \tag{2.4}$$

where $\boldsymbol{x}_0 \in \mathcal{O}$. A curve $\boldsymbol{\alpha} : I \to \mathbb{R}^N$ solves (2.4) *if $\boldsymbol{\alpha}$ solves the first-order system and $\boldsymbol{\alpha}(t_0) = \boldsymbol{x}_0$.*

We will also refer to the initial value problem *generated* by a vector field, meaning the first-order system of differential equations with an arbitrary initial condition, as in definition 2.5.

From practical reasons, it is often more convenient to express the IVP as an *integral equation* by employing the Riemann-integral for vector valued functions (the vector consisting of the Riemann-integral of the component functions cf. [Rudin, 1964, pp. 116-117]).

Proposition 2.2 (Integral representation). *Let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ be continuous. A curve $\boldsymbol{\alpha} : I \to \mathbb{R}^N$ is a solution to the IVP generated by $\boldsymbol{X}$ if, and only if, $\boldsymbol{\alpha} \in C^0$ solves the integral equation*

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{X}(s,\boldsymbol{x}(s))ds, \quad t \in I. \tag{2.5}$$

Proof. First, suppose $\boldsymbol{\alpha}$ is a solution to the IVP generated by $\boldsymbol{X}$. By definition 2.3, since $\boldsymbol{X}$ is continuous on $\mathcal{O}$, $\boldsymbol{\alpha}'$ is continuous on I. This implies that each component function is continuous ([Apostol, 1974, Theorem 4.29]) and, by [Apostol, 1974, Theorem 7.28], $\boldsymbol{\alpha}'$, is integrable on I.

By the fundamental theorem of calculus, [Apostol, 1974, Theorem 7.34], we have (assuming $t \in I$)

$$\begin{aligned}\boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{X}(s,\boldsymbol{\alpha}(s))ds &= \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{\alpha}'(s)ds \\ &= \boldsymbol{x}_0 + (\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(t_0)) \\ &= \boldsymbol{\alpha}(t).\end{aligned}$$

To prove the converse assume that $\boldsymbol{\alpha}$ solves (2.5). Then $\boldsymbol{\alpha} \in C^1$, by [Apostol, 1974, Theorem 7.32]. The properties of the Riemann-integral, cf. [Apostol, 1974, Definition 7.5], gives

$$\boldsymbol{\alpha}(t_0) = \boldsymbol{x}_0 + \int_{t_0}^{t_0} \boldsymbol{X}(s,\boldsymbol{\alpha}(s))ds = \boldsymbol{x}_0.$$

By assumption the left-hand side of (2.5) is differentiable, and applying the fundamental theorem of (integral) calculus, yields

$$\boldsymbol{\alpha}'(t) = \frac{d}{dt}\left(\boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{X}(s,\boldsymbol{\alpha}(s))ds\right) = \boldsymbol{X}(t,\boldsymbol{\alpha}(t)),$$

as desired. □

We refer to the above integral equation as the *integral equation of the IVP*. Although it only applies in the case where the vector field is continuous, this is sufficiently general for our purposes. Our own system with the vector field $\boldsymbol{X}(x,v) = (v,F(x))$ is clearly continuous since by the discussion on p. 2, the force F (given by equation (1.2)) is continuous — and v is differentiable and hence continuous by assumption (it is *introduced* as a differentiable curve, cf. p. 11). The integral equation is useful in connection with existence and uniqueness, partly because it expresses the initial value problem using only one equation, partly because the right-hand side can be viewed as a mapping of continuous curves. Denoting this mapping by T, we can write

$$T(\boldsymbol{x}) = \boldsymbol{x}. \tag{2.6}$$

Thus, existence and uniqueness of solutions of a particular system of differential equations can be reformulated as the problem of finding a unique fixed point of the mapping T, i.e. an $\boldsymbol{x}$ which satisfies (2.6).

CHAPTER 3

Applied Abstract Analysis

In this chapter, our goal will be to develop the basic tools needed in proving existence and uniqueness of solutions to the integral equation of the general IVP. We mentioned in chapter 2 that the integral equation of the IVP defines a mapping of functions — in this chapter we will study the properties of such mappings in a more general setting, and prove that under special circumstances, we can readily conclude that unique fixed points exist.

The methods rely on the concept of sequences in metric spaces, in particular sequences in the underlying metric space of a normed space. The following sections provides an exposition of some fundamental results concerning completeness of normed spaces. The chapter is based on [Apostol, 1974, Section 4.1-4.5, Section 9.1-9.4] (metric spaces and sequences) and [Berg, 1997, Kapitel 5] (Banach spaces and completeness results).

3.1 Sequences in Metric Spaces

We first give a brief review of sequences in general metric spaces. The fundamental concepts of e.g. convergence of sequences and Cauchy sequences form the basis for a study of normed spaces from an analyst's point of view. The discussion, which follows in section 3.2, relies on a precise and general definition of the various properties of such sequences.

Definition 3.1 (Metric space). *A metric space is a non-empty set M (whose elements are called points) together with a function $d : M \times M \to \mathbb{R}$ satisfying the following properties for all points $x, y, z \in M$*

(i) $d(x,y) \geq 0$ with equality if, and only if $x = y$ (positivity);

(ii) $d(x,y) = d(y,x)$ *(symmetry);*

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ *(triangle inequality).*

The function d *is called a metric on* M *and we denote the metric space by* (M,d).

Similarly to sequences in $\mathbb{R}$ and $\mathbb{C}$, we can define sequences in general metric spaces. Using the metric, most of the basic results on convergence of sequences can be reformulated so that they also apply in metric space.

Definition 3.2 (Sequence in metric space). *A sequence in a metric space* (M,d) *is a function* $x : \mathbb{N} \to M$.

Notation. We will use the standard notation x_n for the function x evaluated in n, and write $\{x_n\}_{n\geq 1}$ for the range of x.

Definition 3.3 (Convergent sequence). *A sequence* $\{x_n\}_{n\geq 1}$ *in a metric space* (M,d) *is said to converge to a point* $p \in M$ *if*

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} : n \geq N \Rightarrow d(x_n,p) < \varepsilon.$$

We say that p *is the limit of the sequence* $\{x\}_{n\geq 1}$. *This is written* $x_n \to p$, *or alternatively* $lim_{n\to\infty} x_n = p$.

If the sequence does not converge, it is said to diverge.

A simple application of the triangle inequality shows that the limit of a sequence is unique, cf. [Apostol, 1974, Theorem 4.2].

Definition 3.4 (Cauchy sequence). *A sequence* $\{x_n\}_{n\geq 1}$ *in a metric space* (M,d) *is said to be Cauchy if*

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} : n,m \geq N(\varepsilon) \Rightarrow d(x_n, x_m) < \varepsilon.$$

Proposition 3.1 (Boundedness property). *Every Cauchy sequence in a metric space* (M,d) *is bounded.*

Proof. Let $\{x_n\}_{n\geq 1}$ be Cauchy in M. This means that for all $\varepsilon > 0$ there exists an N such that for all $n \geq N$ we have $d(x_n,x_N) < \varepsilon$. Choose $\varepsilon = 1$ and consider the set

$$\Omega := \{d(x_i,x_j) \mid 1 \leq i,j \leq N\}.$$

Since this set is finite it has a largest member $\omega := \max \Omega$. Application of the triangle inequality now yields, with $1 \leq m \leq N$

$$d(x_n,x_m) \leq d(x_n,x_N) + d(x_m,x_N) \leq 1 + \omega < 2 + \omega.$$

so

$$x_n \in B(x_1, 2 + \omega)$$

for every $n \in \mathbb{N}$. □

Proposition 3.2 (Convergence of Cauchy sequences). *A convergent sequence in a metric space is Cauchy.*

Proof. Let $\{x_n\}_{n \geq 1}$ converge in the metric space (M,d), and denote its limit by p. Then, for all $\varepsilon > 0$ there exists $N(\varepsilon/2)$ such that whenever $n \geq N(\varepsilon/2)$ we have $d(x_n,p) < \varepsilon/2$.

Suppose $n,m \geq N(\varepsilon/2)$. By the triangle inequality we get

$$d(x_m,x_n) \leq d(x_m,p) + d(p,x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This shows that $\{x_n\}_{n \geq 1}$ is a Cauchy sequence. □

Note that the converse of proposition 3.2 is not necessarily true. That is, the Cauchy property is *not* a sufficient condition for convergence. This is in contrast to the equivalent result in $\mathbb{R}$, [Apostol, 1974, Theorem 4.8], and is so important that we define the property of *completeness of a metric space.*

Definition 3.5 (Completeness). *A metric space (M, d) is said to be complete if every Cauchy sequence in (M, d) converges to a point $p \in M$.*

3.1.1 Banach's Fixed Point Theorem

We now present an important result concerning mappings of a complete metric space into itself. This result together with the results on so-called complete normed spaces of the following section will be our primary tool in proving existence and uniqueness for the IVP defined in section 2.2

Definition 3.6 (Contraction mapping). *Let (M,d) be a metric space and suppose $T : M \to M$. Then T is said to be a contraction mapping if*

$$\exists\, \alpha \in]0;1[: \; x,y \in M \Rightarrow d(T(x),T(y)) \leq \alpha d(x,y).$$

The constant α is called a contraction constant.

The following lemma gives a formula for the partial sum of the geometric series which will be used in the proof of theorem 3.1.

Lemma 3.1. *Suppose $\alpha \in \mathbb{R}$, $\alpha \neq 1$. Then*

$$\sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}.$$

Proof. Set

$$s_n := \sum_{k=0}^{n} \alpha^k = 1 + \alpha + \alpha^2 + \cdots + \alpha^n.$$

Then

$$s_n - \alpha s_n = s_n(1-\alpha) = 1 - \alpha^{n+1} \Leftrightarrow s_n = \frac{1-\alpha^{n+1}}{1-\alpha}.$$

□

Theorem 3.1 (Banach's fixed point theorem). *Let (M,d) be a complete metric space and suppose $T : M \to M$ is a contraction mapping. Then T has a unique fixed point x_0.*

Proof. Choose an arbitrary point $x \in M$ and define the sequence $\{a_n\}_{n \geq 0}$ by

$$a_0 := x \text{ and } a_{n+1} := T(a_n), \quad n = 0,1,\ldots$$

We wish to show that the sequence converges and that its limit is in fact a fixed point of T. Since (M,d) is complete, a sufficient condition for the convergence of $\{a_n\}_{n \geq 1}$ is that it satisfies the Cauchy property. We claim that

$$d(a_{n+1},a_n) \leq \alpha^n d(a_1,a_0). \tag{3.1}$$

This is proved by induction. For $n = 1$, (3.1) is obviously true, since by the contraction property of T

$$d(a_2,a_1) \leq \alpha d(a_1,a_0).$$

Suppose 3.1 holds for $n > 1$. By the contraction property and the induction hypothesis, we see that

$$\begin{aligned} d(a_{n+1},a_n) &\leq \alpha d(a_n,a_{n-1}) \\ &\leq \alpha(\alpha^{n-1} d(a_1,a_0)) \\ &= \alpha^n d(a_1,a_0). \end{aligned}$$

Assume $m > n$. Repeated application of the triangle inequality gives us

$$\begin{aligned} d(a_m,a_n) &\leq \sum_{k=n}^{m-1} d(a_{k+1},a_k) \\ &\leq d(a_1,a_0) \sum_{k=n}^{m-1} \alpha^k = d(a_1,a_0) \left(\sum_{k=0}^{m-1} \alpha^k - \sum_{k=0}^{n-1} \alpha^k \right). \end{aligned}$$

Using lemma 3.1, we have

$$\begin{aligned} d(a_m,a_n) &\leq d(a_1,a_0) \left(\frac{1-\alpha^m}{1-\alpha} - \frac{1-\alpha^n}{1-\alpha} \right) \\ &= d(a_1,a_0) \frac{\alpha^n - \alpha^m}{1-\alpha} < \frac{\alpha^n}{1-\alpha} d(a_1,a_0). \end{aligned}$$

But $0 < \alpha < 1$ and therefore $\alpha^n \to 0$ as $n \to \infty$ proving that $\{a_n\}_{n \geq 0}$ is Cauchy. Now since M by assumption is complete, the sequence also converges in M; denote the limit by x_0. Then x_0 is a fixed point for T, since

$$\begin{aligned} d(T(x_0),x_0) &\leq d(T(x_0),a_n) + d(a_n,x_0) \\ &= d(T(x_0),T(a_{n-1})) + d(a_n,x_0) \\ &\leq \alpha d(x_0,a_{n-1}) + d(a_n,x_0), \end{aligned}$$

where the last inequality is due to the contraction property of T. Invoking convergence of $\{a_n\}_{n \geq 1}$ we get

$$d(x_0,a_{n-1}) + d(a_n,x_0) \to 0,$$

which in turn means that $d(T(x_0),x_0) \to 0$ proving that $T(x_0) = x_0$.

This fixed point is indeed unique, since if y_0 were also a fixed point, we would have $d(x_0,y_0) \leq \alpha d(x_0,y_0) \Rightarrow d(x_0,y_0) = 0$ so $x_0 = y_0$. $\square$

3.2 Normed Spaces

We now introduce a slightly less general concept than that of metric spaces, so-called normed (linear) spaces. This will allow us to deal with functions between sets and normed spaces in a much easier and intuitive way than if we used metric spaces solely.

Definition 3.7 (Normed vector space). *A normed space is a vector space* $(V,\mathbf{F},+,)$ *together with a function* $||\cdot|| : V \to \mathbb{R}_0^+$ *(called a norm) satisfying*

(i) $||x|| \geq 0$ *with equality if, and only if* $x = 0$ *(positivity);*

(ii) $||\lambda x|| = |\lambda|\,||x||$ *for all* $\lambda \in \mathbf{F}$ *(homogeneity);*

(iii) $||x + y|| \leq ||x|| + ||y||$ *(triangle inequality).*

Notation. To ease notation, in the following, when we write $(V,||\cdot||)$, we always mean $((V,\mathbb{R},+,),||\cdot||)$ with standard addition and scalar multiplication. Also, if the choice of norm is not important we will just write "V a normed space".

Note that any normed vector space is also a metric space (V,d) when we introduce the function

$$d(x,y) := ||x - y||.$$

This function defines a metric on V. Clearly, positivity and the triangle inequality is fulfilled by this function, since the norm also has these properties. Finally, symmetry is also evident because

$$d(x,y) = ||x - y|| = ||-(-x + y)|| = |-1|\,||y - x|| = ||y - x|| = d(y,x).$$

We say that the metric d is *induced* by the norm $||\cdot||$ on V.

In the following, we will consider the set of functions mapping from a nonempty set S into a a normed space. Defining addition and scalar multiplication

$$\begin{aligned} (f + g)(x) &:= f(x) + g(x); \\ (\lambda f)(x) &:= \lambda f(x), \quad \lambda \in S, \end{aligned}$$

for $x \in S$, turns this set into a vector space. This can be verified simply using the axioms for a vector space see [Axler, 1997, p. 9].

Proposition 3.3. *Given some nonempty set* $S \subseteq \mathbb{R}$*, which has a supremum, and* $\lambda > 0$*, then the set*

$$\tilde{S} = \{\lambda x \mid x \in S\},$$

also has a supremum and $\sup \tilde{S} = \lambda \sup S$.

Proof. Let $a := \sup S$. Choose $x \in \tilde{S}$. Then $x = \lambda y$ for some $y \in S$, so since $x = \lambda y \leq \lambda a$, λa is an upper bound for $\tilde{S}$. Hence $\tilde{S}$ has a supremum — set $b := \sup \tilde{S}$. We see that $b \leq \lambda a$.

To see that λa is in fact the least upper bound, by [Apostol, 1974, Theorem 1.14], for any $\varepsilon > 0$ there exists $z_\varepsilon \in S$ such that $a - \varepsilon < z_\varepsilon$. Multiplying by λ we see that $\lambda a - \lambda\varepsilon < \lambda z_\varepsilon \leq b$ so by [Apostol, 1974, Theorem 1.1], $b \geq \lambda a$ which implies $b = \lambda a$. ☐

Notation. In the following we will write $\sup_{x\in S}||f(x)||$ as short-hand notation for

$$\sup\{||f(x)||\,|\,x\in S\}.$$

Proposition 3.4 (Space of bounded functions). *Let S be a non-empty set, $(V,||\cdot||)$ a normed space and consider the space*

$$\mathcal{B}(S,V) := \{f : S \to V \mid \sup_{x\in S}||f(x)|| < \infty\}.$$

Then the function $||\cdot||_\infty : \mathcal{B}(S,V) \to \mathbb{R}$ given by

$$||f||_\infty := \sup\{||f(x)||\,|\,x\in S\},$$

defines a norm on $\mathcal{B}(S,V)$ called the sup norm.

Proof. To prove this, we must verify that the following conditions for arbitrary functions $f,g \in \mathcal{B}(S,V)$ and $\lambda \in \mathbb{R}$ are satisfied

(i) $||f||_\infty \geq 0$ with equality iff $f = 0$.

(ii) $||\lambda f||_\infty = |\lambda|\,||f||_\infty$.

(iii) $||f+g||_\infty \leq ||f||_\infty + ||g||_\infty$.

(i) Follows by application of the general properties of the norm defined on $\mathcal{B}(S,V)$

$$||f||_\infty = \sup_{x\in S}||f(x)|| > 0 \text{ if } f \neq 0 \text{ and } ||f||_\infty = 0 \text{ if } ||f(x)|| = 0,\ \forall\, x\in S.$$

(ii) Let $\lambda \in \mathbb{R}$. Then we have, by proposition 3.3

$$||\lambda f||_\infty = \sup_{x\in S}||\lambda f(x)|| = \sup_{x\in S}|\lambda|\,||f(x)|| = |\lambda|\sup_{x\in S}||f(x)|| = |\lambda|\,||f||_\infty.$$

(iii) By the properties of the least upper bound, [Apostol, 1974, Theorem 1.14], given any $\varepsilon > 0$ there exists $x_\varepsilon \in S$ such that

$$\sup_{x\in S}||f(x)+g(x)|| - \varepsilon \leq ||f(x_\varepsilon)+g(x_\varepsilon)||.$$

Hence

$$\begin{aligned} ||f+g||_\infty &\leq ||f(x_\varepsilon)+g(x_\varepsilon)|| + \varepsilon \\ &\leq ||f(x_\varepsilon)|| + ||g(x_\varepsilon)|| + \varepsilon \\ &\leq ||f||_\infty + ||g||_\infty + \varepsilon. \end{aligned}$$

Since ε was arbitrarily chosen, by [Apostol, 1974, Theorem 1.1], we have

$$||f+g||_\infty \leq ||f||_\infty + ||g||_\infty.$$

□

When considering sequences of functions, it is necessary to refine the definition of convergence, since for each value of the independent variable, a sequence of functions defines a sequence in its range. If we can apply the usual definition of convergence (definition 3.3) individually for each point in the domain, the sequence of functions is said to converge *pointwise*.

Definition 3.8 (Pointwise convergence). *Let S be a non-empty set, let $(V,||\cdot||)$ be a normed space and suppose $f_n : S \to V$. A sequence of functions $\{f_n\}_{n\geq 1}$ is said to converge pointwise to L if*

$$\forall\, \varepsilon > 0, \forall\, x \in S, \exists\, N(\varepsilon, x) \in \mathbb{N} : n \geq N \Rightarrow \|f_n(x) - L(x)\| < \varepsilon.$$

We write $f_n \xrightarrow{p} L$.

Definition 3.8 states that a sequence of functions converges pointwise to p if, for each point x in the domain, we can find a number N that depends on *both* ε and x, such that the definition of convergence is satisfied. For some sequences of functions we can strengthen this assumption by finding a number N that depends solely on ε; this is called uniform convergence.

Definition 3.9 (Uniform convergence). *Let S be a non-empty set, let $(V,||\cdot||)$ be a normed space and suppose $f_n : S \to V$. A sequence of functions $\{f_n\}_{n\geq 1}$ is said to converge uniformly to L if*

$$\forall\, \varepsilon > 0, \exists\, N(\varepsilon) \in \mathbb{N} : n \geq N(\varepsilon) \wedge x \in S \Rightarrow \|f_n(x) - L(x)\| < \varepsilon.$$

We write $f_n \xrightarrow{u} L$.

Observe that uniform convergence implies pointwise convergence.

Uniform convergence is closely connected with convergence of sequences in the space of functions when endowed with the sup norm (introduced in proposition 3.4). In fact, convergence in the function space is equivalent to uniform convergence of the corresponding sequence of functions.

Theorem 3.2 (Convergence in sup norm). *$f_n \to L$ in $(\mathcal{B}(S,V),||\cdot||_\infty)$ if, and only if, $f_n \xrightarrow{u} L$.*

Proof. Suppose $f_n \xrightarrow{u} L$, that is

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} : n \geq N(\epsilon) \wedge x \in S \Rightarrow ||f_n(x) - L(x)|| < \varepsilon.$$

Choose $\varepsilon' := \varepsilon/2$ in the definition of uniform convergence. Since the convergence does not depend on x (it is uniform), we conclude that for $n \geq N(\varepsilon')$, the set $\{||f_n(x) - L(x)||\}_{n \geq N(\varepsilon')}$ is bounded above by ε' and therefore

$$||f - L||_\infty = \sup_{x \in S} ||f_n(x) - L(x)|| \leq \varepsilon' < \varepsilon.$$

To prove the converse, assume $f_n \to L$ in $(\mathcal{B}(S,V), ||\cdot||_\infty)$. Formally this means

$$\forall \varepsilon > 0, \exists N(\varepsilon) : n \geq N \Rightarrow \sup_{x \in S} ||f_n(x) - L(x)|| < \varepsilon.$$

With these choices of ε and $N(\varepsilon)$, by the definition of the least upper bound, for every $x \in S$ we must have

$$||f_n(x) - L(x)|| \leq \sup_{x \in S} ||f_n(x) - L(x)|| = ||f_n - L||_\infty < \varepsilon,$$

Therefore $f_n \xrightarrow{u} L$. □

3.2.1 Completeness of $(\mathcal{B}(S,V), ||\cdot||_\infty)$

In this section we will show that a certain normed space of bounded functions, with respect to the sup norm has the property that the underlying metric space is complete. Such complete normed spaces are called *Banach spaces.*

Definition 3.10 (Banach space). *A Banach space is a normed space, which is complete with respect to the metric induced by the norm.*

In particular, we will prove that completeness of $(\mathcal{B}(S,V), ||\cdot||_\infty)$ implies that the subspace of continuous (bounded) functions is also a Banach space. This is important, since it will allow us to apply Banach's fixed point theorem on mappings (operators) defined on this space — the integral operator, for instance, acts on such spaces of continuous functions.

We will need the following lemma.

Lemma 3.2. *Let* $\{x_n\}_{n \geq 1}$ *be a sequence in a normed space* $(V, ||\cdot||)$*. Then*

$$x_n \to x \Rightarrow ||x_n|| \to ||x||.$$

Proof. Assume $x_n \to x$, that is

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} : n \geq N(\varepsilon) \Rightarrow ||x_n - x|| < \varepsilon.$$

Applying the triangle inequality on $x_n = x_n - x + x$ yields

$$||x_n|| \leq ||x_n - x|| + ||x|| \Leftrightarrow ||x_n|| - ||x|| \leq ||x_n - x||,$$

similarly for $x = x - x_n + x_n$ we have

$$||x|| \leq ||x - x_n|| + ||x_n|| \Leftrightarrow -||x_n - x|| \leq ||x_n|| - ||x||.$$

so $|\, ||x_n|| - ||x|| \,| \leq ||x_n - x||$. In particular, for $n \geq N(\varepsilon)$, we have

$$|\, ||x_n|| - ||x|| \,| \leq ||x_n - x|| < \varepsilon,$$

and therefore $||x_n|| \to ||x||$. □

We are now in a position to state the main theorem of this section.

Theorem 3.3 (Completeness of $(\mathcal{B}(S,V),||\cdot||_\infty)$). *If $(V,||\cdot||)$ is a Banach space, then $(\mathcal{B}(S,V),||\cdot||_\infty)$ is also a Banach space.*

Proof. We must verify that the underlying metric space with metric induced by the sup norm is complete — in other words, we must prove that every Cauchy sequence in $(\mathcal{B}(S,V),||\cdot||_\infty)$ converges.

This will be proved assuming that $\{f_n\}_{n\geq 1} \in \mathcal{B}(S,V)$ is Cauchy, and showing that this implies

(i) $\{f_n\}_{n\geq 1}$ converges pointwise, $f_n \xrightarrow{p} L$.

(ii) $||L||_\infty < \infty$ and $f_n \xrightarrow{u} L$. By Theorem 3.2, this is equivalent to

$$f_n \to L \text{ in } (\mathcal{B}(S,V),||\cdot||_\infty).$$

Assume the sequence $\{f_n\}_{n\geq 1}$ is Cauchy in $(\mathcal{B}(S,V),||\cdot||_\infty)$, that is for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$||f_n - f_m||_\infty < \varepsilon, \quad \text{whenever} \quad n,m \geq N.$$

Note that the set of Cauchy sequences is nonempty, since every constant function mapping from S to V is clearly Cauchy (and bounded).

The Cauchy property implies that for fixed x, $\{f_n(x)\}_{n\geq 1}$ is also Cauchy since for all $x \in S$

$$||f_n(x) - f_m(x)|| \leq \sup_{x\in S} ||f_n(x) - f_m(x)|| < \varepsilon.$$

By completeness of $(V,||\cdot||)$, the (pointwise) limit $\lim_{n\to\infty} f_n(x)$ exists for every $x \in S$. Denote this limit by L, and define the function $x \mapsto L(x)$. This function is also an element of $(\mathcal{B}(S,V),||\cdot||_\infty)$. This follows since we assumed that $\{f_n\}_{n\geq 1}$ was Cauchy, so by proposition 3.1, it is also bounded i.e.

$$\exists\, M \geq 0 : ||f_n||_\infty \leq M, \quad \forall\, n \in \mathbb{N}.$$

Choose $x \in S$, and apply the definition of pointwise convergence.

$$\forall\, \varepsilon > 0, \exists\, N(\varepsilon,x) \in \mathbb{N} \,:\, n \geq N(\varepsilon,x) \Rightarrow ||f_n(x) - L(x)|| < \varepsilon.$$

Choosing $\varepsilon := 1$ and $N' := N(1,x)$ yields

$$||L(x)|| = ||L(x) + f_{N'}(x) - f_{N'}(x)|| \leq ||L(x) - f_{N'}(x)|| + ||f_{N'}(x)|| < 1 + M.$$

Since x was arbitrarily chosen, L is bounded.

We now use the pointwise limit function to show that $f_n \xrightarrow{u} L$. Since $\{f_n\}_{n\geq 1}$ is Cauchy, we can let $\varepsilon > 0$ be given and choose an integer $N(\varepsilon/2)$ such that for every $x \in S$

$$||f_n(x) - f_m(x)|| < \frac{\varepsilon}{2}, \quad \text{whenever } n,m \geq N\left(\frac{\varepsilon}{2}\right).$$

Fix n and consider the sequence $\{||f_n(x) - f_m(x)||\}_{m\geq n}$. By lemma 3.2, since $f_m \xrightarrow{p} L$, this sequence converges

$$\lim_{m\to\infty} ||f_n(x) - f_m(x)|| = ||f_n(x) - L(x)|| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Take notice of the above. The assumption $||f_n(x) - f_m(x)|| < \varepsilon/2$ is *not* sufficient to conclude

$$\lim_{m\to\infty} ||f_n(x) - f_m(x)|| < \frac{\varepsilon}{2}.$$

However, the limit must adhere to the set, in which the terms of the sequence lie and by [Berg, 1997, Sætning 2.10], the set of adherent points is exactly the closure of the set. An equivalent result is stated in lemma 3.3.

This shows that $f_n \xrightarrow{u} L$ and therefore $f_n \to L$ in $(\mathcal{B}(S,V),||\cdot||_\infty)$. $\square$

The central idea in the above proof — namely that uniform convergence of sequences of functions implies convergence in the function space — clearly suggests that the completeness result is intimately connected with our choice of norm for $\mathcal{B}(S,V)$.

Indeed, in infinite dimensional normed spaces there generally exists many non-equivalent norms; this meaning that a sequence which converges in one norm not necessarily converges in another norm. In section 3.2.2, we will prove that for finite dimensional normed spaces all norms are equivalent, a result which is used extensively throughout the rest of the report. A more formal statement of equivalence of norms says that equivalent norms generate the same *topology*, [Rudin, 1987, Definition 1.2].

Lemma 3.3. *A subset S of a normed space $(V,||\cdot||)$ is closed if, and only if, every sequence in S, which is convergent in V, has a limit in S.*

Proof. Assume S is closed. Let $\{x_n\}_{n\geq 1}$ be a sequence in S and suppose that $x_n \to x$, $x \notin S$. Now, the set $V\backslash S$ is by definition open and therefore

$$\exists\, \varepsilon > 0 : B(x,\varepsilon) \subseteq V\backslash S.$$

But since $x_n \to x$, we also have

$$\exists\, N(\varepsilon) \in \mathbb{N} : n \geq N \Rightarrow ||x_n - x|| < \varepsilon.$$

This is a contradiction, since $\{x_n\}_{n\geq 1} \subseteq S$ and we conclude that $x \in S$.

Conversely, assume that every sequence in S has a limit in S. We wish to show that this implies that S is closed. Assume the contrary, namely that S is not closed. Then $V\backslash S$ is not open which implies that there exists $x \in V\backslash S$ such that

$$\forall\, \varepsilon > 0, B(x,\varepsilon) \cap S \neq \emptyset.$$

Clearly, we can construct a sequence $\{x_n\}_{n\geq 1}$ in S such that $x_n \in B(x,1/n)$. However, the limit of this sequence must be x which is a contradiction since we assumed $x \in V\backslash S$. We conclude that S is closed. $\square$

An equivalent result applies in general metric spaces; a metric space is closed iff it contains all its adherent points, cf. [Apostol, 1974, Theorem 3.37].

The following is a straightforward consequence of lemma 3.3 but very useful when characterising subspaces of a Banach space. We will use the result in theorem 3.4.

Lemma 3.4. *Suppose V is a Banach space. Then every closed subset H of V is a complete metric space with respect to the induced metric.*

Proof. Suppose a sequence is Cauchy in the subspace H. Then the sequence is Cauchy in $(V,||\cdot||)$ and because of completeness it is convergent. However, by lemma 3.3, since H is closed, the limit must lie in H. □

Theorem 3.4. *Suppose $(V,||\cdot||_V)$ is a normed space and $S \subseteq V$ is compact. Let $(W,||\cdot||)$ be a Banach space and define*

$$\mathcal{C}(S,W) := \{f : S \to W \mid f \in C^0\}.$$

Then $(\mathcal{C}(S,W),||\cdot||_\infty)$ is a Banach space.

Proof. By [Apostol, 1974, Theorem 4.25], compactness of S implies compactness of $f(S)$ for every $f \in \mathcal{C}(S,W)$ (since f by assumption is continuous). Thus for every $f \in \mathcal{C}(S,W)$ there exists $B \geq 0$ such that $||f||_\infty \leq B$.

This implies that $(\mathcal{C}(S,W),||\cdot||_\infty)$ is a subspace of $(\mathcal{B}(S,W),||\cdot||_\infty)$ (using standard addition and scalar multiplication of functions).

In order to show completeness of this subspace, we prove that $(\mathcal{C}(S,W),||\cdot||_\infty)$ is closed and apply lemma 3.4.

By lemma 3.3 a subset of a normed space is closed if and only if it contains all limits of sequences in the subset. Suppose the sequence $\{f_n\}_{n\geq 1} \in \mathcal{C}(S,W)$ is convergent in $(\mathcal{B}(S,W),||\cdot||_\infty)$, and denote its limit by L.

Since this is equivalent to uniform convergence, cf. theorem 3.2, choose $\varepsilon/3$ and $N = \tilde{N}(\varepsilon/3) \in \mathbb{N}$ such that for every $n \geq N$ and every $x \in S$ we have

$$||f_n(x) - L(x)|| < \frac{\varepsilon}{3}.$$

Choose $n = N$. We now employ the fact that f_N is continuous on S, that is for every $x_0 \in S$, we can for ε find a $\delta > 0$ such that

$$||x - x_0||_V < \delta \Rightarrow ||f_N(x) - f_N(x_0)|| < \frac{\varepsilon}{3}.$$

Applying the triangle inequality yields the following

$$\begin{aligned} ||L(x) - L(x_0)|| &\leq ||L(x) - f_N(x)|| + ||f_N(x) - f_N(x_0)|| + ||f_N(x_0) - L(x_0)|| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Thus L is continuous on S, so we conclude $L \in \mathcal{C}(S,W)$. Since $\{f_n\}_{n\geq 1}$ was arbitrarily chosen, by lemma 3.3, $\mathcal{C}(S,W)$ is closed and applying lemma 3.4 proves completeness. □

The above theorem is just one way of stating the general result that if a sequence of continuous functions converges uniformly, then the limit is also a continuous function.

Remark. It is possible to prove theorem 3.4 in a more general setting where one considers the normed space of continuous, bounded functions mapping from a *topological space* (see [Rudin, 1987, Definition 1.2]) into a Banach space. This is because continuity of functions is a topological property, cf. [Berg, 1997, p. 31]. The above is just a special case where the topological space is a normed space and we consider a bounded subset of this normed space in order to achieve the boundedness property. This will suffice for our purposes.

According to [Berg, 1997, p. 1.4], $(\mathbb{R}^N, \ell_2)$ is a normed space (ℓ_2 being the Euclidean norm). This allows us to state the above result for the special case concerning *vector-valued continuous functions.*

Corollary 3.1. *Suppose I is a closed interval. Then $(\mathcal{C}(I,\mathbb{R}^N),||\cdot||_\infty)$ is complete.*

Proof. Since I is a closed (and bounded) interval it is compact [Apostol, 1974, Theorem 3.31]. From [Apostol, 1974, Theorem 4.8], we know that every Cauchy sequence converges in $\mathbb{R}^N$, hence $\mathbb{R}^N$ is complete. □

The following proposition, will be of great importance throughout the chapter of existence and uniqueness. It states a slightly less general result for the case where continuous functions are mapped from a compact subset of a normed space into a *subset* of a Banach space. We cannot generally impose vector space structure on this space, but it will always be a complete metric space.

Proposition 3.5. *Suppose $(V,||\cdot||_V)$ is a normed space and $S \subseteq V$ is compact. Let $(W,||\cdot||)$ be a Banach space and assume R is a closed subset of W. Then $\mathcal{C}(S,R)$ is a complete metric spaces under the induced metric $d_\infty(f,g) := ||f-g||_\infty$, $f,g \in \mathcal{C}(S,R)$.*

Proof. According to lemma 3.4, we should show that $(\mathcal{C}(S,R),d_\infty)$ is a closed subset of the Banach space $(\mathcal{C}(S,W),||\cdot||_\infty)$.

Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{C}(S,R),d_\infty)$. Because of completeness of $(\mathcal{C}(S,W),||\cdot||_\infty)$, the sequence of functions converges to a limit L in this space. According to theorem 3.2 this is equivalent to uniform convergence, i.e. $f_n \xrightarrow{u} L$. So, for each $x \in S$ we have the (pointwise) limit $f_n(x) \to L(x)$; since R is a closed subset of W, we also have $L(x) \in R$, by lemma 3.3.

Therefore, we conclude that $L : S \to R$, and since L by assumption is continuous (it is also a limit in $(\mathcal{C}(S,W),||\cdot||_\infty)$), $(\mathcal{C}(S,R),d_\infty)$ is closed. $\square$

Our strategy for the following chapter on existence and uniqueness of solutions to initial value problems is basically to show that the mapping defined by the integral equation of the IVP acts as an operator on a certain closed subset of the Banach space $(\mathcal{C}(I,\mathbb{R}^n),||\cdot||_\infty)$. It is sufficient to prove that this operator defines a contraction mapping on this subset since existence and uniqueness is then an immediate consequence of Banach's fixed point theorem, theorem 3.1.

3.2.2 Equivalence of Norms on Finite Dimensional Spaces

It is a fundamental result on finite dimensional normed spaces that in fact every norm induces the same topology on the space. Briefly explained, this implies the basic property of convergence is left intact when using another norm and hence we can freely choose any norm when investigating convergence of sequences, continuity etc. We will exploit this fact in many connections and therefore the following section is dedicated to showing equivalence of norms on finite dimensional spaces and proving that convergence in one norm implies convergence in any norm.

Definition 3.11. *Let V be a normed space and suppose $||\cdot||_a$ and $||\cdot||_b$ are two norms on V. Then $||\cdot||_a$ and $||\cdot||_b$ are said to be equivalent if*

$$\exists\, C_1,C_2 > 0\,,:\; C_1||v||_a \le ||v||_b \le C_2||v||_a, \quad \forall\, v \in V. \tag{3.2}$$

Equivalence of norms defines an equivalence relation (cf. [Rudin, 1964, Remark in Definition 2.5], setting

$$||\cdot||_a \;\sim\; ||\cdot||_b,$$

if (3.2) holds. Therefore the result of this section is that the set of norms on finite dimensional normed spaces contains only one equivalence class w.r.t. this relation.

Lemma 3.5 (Continuity of norm). *If $(V,||\cdot||)$ is a normed space, then the norm $||\cdot|| : V \to \mathbb{R}_0^+$ is uniformly continuous (on V).*

Proof. We apply the definition of continuity, [Apostol, 1974, Theorem 4.15] using the induced metric $d(x,y) := ||x - y||$ for $x,y \in V$.

Fix $p \in V$ and choose $\varepsilon > 0$. Let $\delta := \varepsilon$ and suppose $||x - p|| < \delta$. Then, by the triangle inequality, we have $|\,||x|| - ||p||\,| \leq ||x - p|| < \varepsilon$. Since p was arbitrarily chosen, we conclude that $||\cdot||$ is uniformly continuous. $\square$

Theorem 3.5 (Equivalence of norms). *Any two norms on a finite dimensional normed space are equivalent.*

Proof. Suppose V is a finite dimensional space and let $||\cdot||_a$ and $||\cdot||_b$ be two different norms on V

Since $\dim V < \infty$, by [Axler, 1997, Corollary 2.11], V has a basis; say $(v_1,v_2,\ldots,v_n)$. Equivalence is trivial whenever $v = 0$, so let $v \in V$ be different from the zero vector

$$v = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n, \quad \mu_j \in \mathbf{F} \text{ for } j = 1,2,\ldots,n,$$

where not all $\mu_j = 0$.

We may assume w.l.o.g. that $\mu_1 \neq 0$. Using the homogeneity of the norm (cf. definition 3.7), we can write

$$\begin{aligned}
\frac{||v||_a}{||v||_b} &= \frac{||\mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n||_a}{||\mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n||_b} \\
&= \frac{|\mu_1|\,||v_1 + (\mu_2/\mu_1)v_2 + \cdots + (\mu_n/\mu_1)v_n||_a}{|\mu_1|\,||v_1 + (\mu_2/\mu_1)v_2 + \cdots + (\mu_n/\mu_1)v_n||_b} \\
&:= \frac{||v_1 + \lambda_1 v_2 + \cdots + \lambda_{n-1} v_n||_a}{||v_1 + \lambda_1 v_2 + \cdots + \lambda_{n-1} v_n||_b}.
\end{aligned}$$

Define the function $F : \mathbf{F}^{n-1} \to \mathbb{R}$ by

$$F(\lambda_1,\ldots,\lambda_{n-1}) := \frac{||v||_a}{||v||_b} = \frac{||v_1 + \lambda_1 v_2 + \cdots + \lambda_{n-1} v_n||_a}{||v_1 + \lambda_1 v_2 + \cdots + \lambda_{n-1} v_n||_b}. \tag{3.3}$$

Denote the vector $(\lambda_1,\ldots,\lambda_{n-1})$ in the natural basis of $\mathbf{F}^{n-1}$ by λ, and endow this space with the ℓ_1-norm. Observe that F is well-defined and strictly positive — since $||v||_{a,b} = 0 \Leftrightarrow v = 0$, by definition, so

$$v_1 + \lambda_1 v_2 + \cdots + \lambda_{n-1} v_n = 0,$$

which implies that $(v_1,\ldots,v_n)$ is a linearly dependent set, contrary to our choice of v_i, $i = 1,2,\ldots,n$ as a basis for V.

Now, for $||\cdot||_a$ and $||\cdot||_b$ to be equivalent, we must have $\sup \operatorname{range} F < \infty$ and $\inf \operatorname{range} F > 0$ since

$$\sup \operatorname{range} F < \infty \Rightarrow \exists\, C_1 > 0 \,:\, ||v||_a/||v||_b \leq C_1 \Leftrightarrow ||v||_a \leq C_1||v||_b,$$

and

$$\inf \operatorname{range} F > 0 \Rightarrow \exists\, C_2 > 0 \,:\, ||v||_a/||v||_b \geq C_2 \Leftrightarrow ||v||_a \geq C_2||v||_b$$

The last demand is not trivially fulfilled although F is strictly positive for all $\lambda \in \mathbf{F}^{n-1}$; 0 could still adhere to the set $\operatorname{range} F$.

However, it is sufficient to show that $\sup \operatorname{range} F < \infty$ holds, since with our our choice of F, $\sup \operatorname{range} F < \infty \Rightarrow \inf \operatorname{range} F > 0$. To see this, define

$$G(\lambda) := \frac{1}{F(\lambda)}$$

The function G is well-defined, since F has been shown to be strictly positive. Also, because the labelling of norms does not matter, we have by assumption $\sup \operatorname{range} G < \infty$. So we should show that

$$\sup \operatorname{range} G = M < \infty \Rightarrow \inf \operatorname{range} F > 0 \tag{3.4}$$

We prove that the contrapositive holds. So assume that $\inf \operatorname{range} = 0$. Since the infimum of a set which has a lower bound, is an adherent point for the set (this follows from [Apostol, 1974, Theorem 1.14] and the definition of adherence), by [Apostol, 1974, Theorem 4.4], there exists a sequence $\{x_n\}_{n\geq 1}$ with $x_i \in \mathbf{F}^{n-1}$, $i = 1,2,\ldots$ such that $F(x_n) \to 0$, i.e.

$$\forall\, \varepsilon > 0,\, \exists\, N(\varepsilon) \in \mathbb{N} \,:\, n \geq N(\varepsilon) \Rightarrow |F(x_n)| < \varepsilon. \tag{3.5}$$

Given any $M > 0$, choose $\varepsilon' := 1/(M+1)$ and pick $n \geq N(\varepsilon')$ so that (3.5) holds with ε'. Then

$$|G(x_n)| > \left|\frac{1}{1/(M+1)}\right| = M + 1 > M.$$

So $G(x_n)$ diverges proving that $\sup \operatorname{range} G = \infty$, hence (3.4) holds and it suffices to show that $\sup \operatorname{range} F < \infty$.

We first show that F defines a continuous function. Now, F is in fact the composite function

$$\lambda \mapsto v_1 + \sum_{i=2}^{n} \lambda_{i-1}v_i \mapsto \left\| v_1 + \sum_{i=2}^{n} \lambda_{i-1}v_i \right\|.$$

Composition of continuous functions yields continuous functions, cf. [Apostol, 1974, Theorem 4.17] and since $||\cdot||$ by lemma 3.5 is continuous, it we must show that the mapping $\lambda \mapsto v_1 + \sum_{i=2} \lambda_{i-1} v_i$ is also continuous. Assume $\lambda \in \mathbf{F}^{n-1}$ and let $||\cdot||_V$ be any norm on V. Set $M := \max_{2\leq i\leq n} ||v_i||_V$ and let $\varepsilon > 0$ be given. Choose $\delta := \varepsilon/M$ and suppose for some $\varphi \in \mathbf{F}^{n-1}$, we have $||\lambda - \varphi|| < \delta$. We then see that

$$\begin{aligned}\left\|(v_1 + \sum_{i=2}^{N} \lambda_{i-1} v_i) - (v_1 + \sum_{i=2}^{N} \varphi_{i-1} v_i)\right\|_V &\leq \left\|\sum_{i=2}^{N} (\lambda_i - \varphi_i) v_i\right\|_V \\ &\leq M \sum_{i=1}^{n-1} |\lambda_i - \varphi_i| \\ &= M||\lambda - \varphi|| < M\delta = \varepsilon.\end{aligned}$$

Hence F is continuous. According to [Apostol, 1974, Theorem 4.27], if we restrict F to the compact ball, $\lambda \in \overline{B_{\mathbf{F}^{n-1}}(0,R)}$, $R > 0$, then the image of F is also bounded subset of $\mathbb{R}$; in this case we readily conclude that $\sup F(\overline{B_{\mathbf{F}^{n-1}}(0,R)})$ is an upper bound for F.

Now, assume $||\lambda|| > R$. By the triangle inequality we have

$$||v_1 + \lambda_1 v_2 + \cdots + \lambda_{n-1} v_n||_a \leq ||v_1||_a + |\lambda_1|\, ||v_2||_a + \cdots + |\lambda_{n-1}|\, ||v_n||_a.$$

Take $m_a := \max_{2\leq i\leq n} ||v_i||_a$. Then

$$||v_1 + \lambda_1 v_2 + \cdots + \lambda_{n-1} v_n||_a \leq ||v_1||_a + m_a ||\lambda||.$$

In order to obtain an estimate for $||\cdot||_b$, we define the following function

$$f : \overline{B_{\mathbf{F}^{n-1}}(0,1)} \to \mathbb{R} \text{ with } f(x) := ||x_1 v_2 + \cdots + x_{n-1} v_n||_b,$$

By the same arguments given for F, f is a strictly positive, continuous function and since it is defined on a compact set, it attains a minimum value (cf. [Apostol, 1974, Theorem 4.28]), i.e.

$$\exists\, x_0 \in \overline{B_{\mathbf{F}^{n-1}}(0,1)} \;:\; f(x_0) = \min_x f(x) := m_b.$$

Now let $x = \lambda/||\lambda|| \in \overline{B_{\mathbf{F}^{n-1}}(0,1)}$. We then see that

$$\begin{aligned}&||v_1 + \lambda_1 v_2 + \cdots + \lambda_{n-1} v_n||_b = \\ &\qquad = ||\lambda||(||v_1/||\lambda|| + x_1 v_2 + \cdots + x_{n-1} v_n||_b) \\ &\qquad \geq ||\lambda||(||x_1 v_2 + \cdots + x_{n-1} v_n||_b - ||v_1||_b/||\lambda||) \\ &\qquad \geq ||\lambda|| m_b - ||v_1||_b,\end{aligned}$$

where we have applied the triangle inequality and used that m_b is the minimum value of f.

Since the ratio in F increases, if we substitute respectively something larger and something smaller in the enumerator and denumerator, we have

$$\begin{aligned} F(\lambda) &\leq \frac{||v_1||_a + m_a||\lambda||}{m_b||\lambda|| - ||v_1||_b} \\ &= \frac{||v_1||_a/||\lambda|| + m_a}{m_b - ||v_1||_b/||\lambda||}. \end{aligned}$$

By assumption $||\lambda|| > R$, so we can make the following estimate

$$F(\lambda) \leq \frac{||v_1||_a/R + m_a}{m_b - ||v_1||_b/R}.$$

Since $\lim_{R\to\infty} 1/R = 0$, we see that

$$\exists\, R_0 > 0 \,:\, ||\lambda|| > R_0 \Rightarrow F(\lambda_1, \ldots, \lambda_{n-1}) \leq 2\frac{m_a}{m_b}.$$

For this choice of R_0, we may conclude that

$$F(\lambda) \leq \max\left\{\sup F(\overline{B(0,R_0)}), 2\frac{m_a}{m_b}\right\} < \infty, \quad \forall\, \lambda \in \mathbf{F}^{n-1}.$$

This completes the proof.

□

Proposition 3.6. *If a sequence in a finite dimensional normed space converges in one norm then it converges in any norm.*

Proof. Let $\{x_n\}_{n\geq 1}$ be a sequence in a finite dimensional normed space V and suppose that $\{x_n\}_{n\geq 1}$ converges to a limit $p \in V$ in some norm $||\cdot||_V$, i.e.

$$\forall\, \varepsilon > 0\,, \exists\, N(\varepsilon) \in \mathbb{N} \,:\, n \geq N(\varepsilon) \Rightarrow ||x_n - p||_V < \varepsilon.$$

Let $||\cdot||$ be another norm on V. Applying to theorem 3.5, there exists $C > 0$ such that $||v|| \leq C||v||_V$ for all $v \in V$. Let $\varepsilon' := \varepsilon/C$, and choose $n \geq N(\varepsilon')$. We immediately see that

$$||x_n - p|| \leq C||x_n - p||_V < C\varepsilon' = \varepsilon.$$

□

CHAPTER 4

Existence and Uniqueness of Solutions

As mentioned earlier, it is a fundamental question whether solutions to a given first-order system exists, and if these solutions are uniquely determined. In this chapter, we will investigate existence and uniqueness, primarily based on the results on convergence of sequences in Banach spaces and metric spaces, discussed in chapter 3, specifically section 3.2.1. These results allow us to deduce local existence and uniqueness under relatively mild assumptions — and it will be shown that a locally, unique solution can always be extended to a unique solution defined on the largest possible interval of times.

These results form the basis for the main theorem of this chapter, that our own system has uniquely determined solutions regardless of initial conditions for all positive times.

4.1 Lipschitz Condition

Although local existence of solutions to an IVP generated by a vector field $\boldsymbol{X}(t,\boldsymbol{x})$ can be proven with no other assumptions on $\boldsymbol{X}$ than continuity (cf. [Po-Fang and Sibuya, 1999, Theorem I-2-5]), it is necessary to impose a stronger condition on $\boldsymbol{X}$ in order to ensure uniqueness. In this report we employ a *sufficient* condition for existence and uniqueness that gives a relatively straightforward proof. This condition is called a *Lipschitz condition.*

Definition 4.1 (Lipschitz condition). *A mapping $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^M$ is said to be Lipschitz (continuous), or to satisfy a Lipschitz condition on a set*

$\Omega \subseteq \mathbb{R}^N$, *if*

$$\exists\, L > 0 \,:\, \boldsymbol{x}, \boldsymbol{y} \in \Omega \Rightarrow ||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})|| \leq L||\boldsymbol{x} - \boldsymbol{y}||,$$

where L *is called a Lipschitz constant.*

A map is said to be globally Lipschitz if $\Omega = \mathbb{R}^N$. *A map is said to be locally Lipschitz if it is Lipschitz in a compact neighbourhood around each* $\boldsymbol{x} \in \mathbb{R}^N$, *i.e. there exists* L,$r > 0$ *such that*

$$\boldsymbol{y}, \boldsymbol{z} \in \overline{B(\boldsymbol{x}, r)} \Rightarrow ||\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{z})|| \leq L||\boldsymbol{y} - \boldsymbol{z}||.$$

Lipschitz continuity can also be defined in general metric spaces, cf. [Berg, 1997, p. 32]

Notation. Let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ be a non-autonomous vector field on $\mathcal{O} \subseteq \mathbb{R}\times\mathbb{R}^N$. We will use the term Lipschitz in the second variable if

$$\exists\, L > 0 \,:\, (t,\boldsymbol{x}), (t,\boldsymbol{y}) \in \Omega \Rightarrow ||\boldsymbol{X}(t,\boldsymbol{x}) - \boldsymbol{X}(t,\boldsymbol{y})|| \leq L||\boldsymbol{x} - \boldsymbol{y}||,$$

i.e. it satisfies a Lipschitz condition in $\boldsymbol{x}$, independently of t.

Remark. Note that if a mapping satisfies a Lipschitz condition with respect to one norm, it satisfies a Lipschitz condition with respect to *any* norm. This follows directly from equivalence of norms on finite dimensional spaces, theorem 3.5; they differ by a constant.

To see this suppose $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^M$ is Lipschitz in $\Omega \subseteq \mathbb{R}^N$ and fix a Lipschitz constant L. Let $||\cdot||_a$ and $||\cdot||_b$ be norms on $\mathbb{R}^N$, i.e. there exists C_1,$C_2 > 0$ such that

$$||\boldsymbol{x}||_a \leq C_1||\boldsymbol{x}||_b \quad \text{and} \quad ||\boldsymbol{x}||_b \leq C_2||\boldsymbol{x}||_a$$

for every $\boldsymbol{x} \in \mathbb{R}^N$, and let $||\cdot||$ be a norm on $\mathbb{R}^M$. Actually, we should consider two norms on $\mathbb{R}^M$ as well, but this is similar.

Now, if $\boldsymbol{f}$ is Lipschitz with respect to $||\cdot||_a$, we have

$$\begin{aligned}\frac{1}{C_2}||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})||_b \leq ||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})||_a \leq L||\boldsymbol{x} - \boldsymbol{y}|| \Rightarrow \\ ||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})||_b \leq LC_2||\boldsymbol{x} - \boldsymbol{y}||,\end{aligned}$$

and since $||\cdot||_a$ and $||\cdot||_b$ were chosen arbitrarily, we see that $\boldsymbol{f}$ is Lipschitz in any norm.

To get a feeling of, how strong a local Lipschitz condition is, let the (autonomous) vector field $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ with $\mathcal{O} \subseteq \mathbb{R}^N$ open be given. Then the following holds:

(i) $\boldsymbol{X}$ is continuous and all partial derivatives

$$\frac{\partial \boldsymbol{X}}{\partial x_1}, \frac{\partial \boldsymbol{X}}{\partial x_2}, \ldots, \frac{\partial \boldsymbol{X}}{\partial x_N},$$

exist and are continuous $\Rightarrow$ $\boldsymbol{X}$ satisfies a local Lipschitz condition.

(ii) $\boldsymbol{X}$ satisfies a local Lipschitz condition in $\boldsymbol{p} \in \mathcal{O} \Rightarrow$ continuity in $\boldsymbol{p}$.†

The last claim is straightforward to verify. Suppose that $\boldsymbol{X}$ satisfies a local Lipschitz condition around $\boldsymbol{p}$

$$\exists\, L,r > 0, :\ \boldsymbol{x},\boldsymbol{y} \in \overline{B(\boldsymbol{p},r)} \Rightarrow ||\boldsymbol{X}(\boldsymbol{x}) - \boldsymbol{X}(\boldsymbol{y})|| \leq L||\boldsymbol{x} - \boldsymbol{y}||.$$

Fix L and let $\varepsilon > 0$ be given. Define $\delta := \min\{r,\varepsilon/2L\}$; then, choosing $\boldsymbol{x},\boldsymbol{y} \in B(\boldsymbol{p},\delta)$ and invoking the Lipschitz condition yields

$$||\boldsymbol{X}(\boldsymbol{x}) - \boldsymbol{X}(\boldsymbol{y})|| \leq L||\boldsymbol{x} - \boldsymbol{y}|| \leq L(||\boldsymbol{x} - \boldsymbol{p}|| + ||\boldsymbol{p} - \boldsymbol{y}||) < 2L\delta \leq \varepsilon.$$

So $\boldsymbol{X}$ is continuous. For the other claim, see [Berg, 1997, Lemma 7.13]. Hence, for autonomous vector fields, local Lipschitz continuity is stronger than continuity but weaker than continuous differentiability.

4.2 Local Existence and Uniqueness

We first present a basic identity for Riemann-integrals of vector valued functions, limited to continuous vector valued functions which will suffice for our purpose. Since we will be dealing with mappings defined by integrals of vector functions on several occasions, this identity is very important for our further work.

Proposition 4.1. *Let I be a closed interval and suppose $\boldsymbol{f} : I \to \mathbb{R}^N$ is continuous. Then $\boldsymbol{f} \in \mathcal{R}(I)$, and for any norm $||\cdot||$ on $\mathbb{R}^N$, $||\boldsymbol{f}|| \in \mathcal{R}(I)$ and*

$$\left\| \int_I \boldsymbol{f}(t)dt \right\| \leq \int_I ||\boldsymbol{f}(t)||dt.$$

Proof. By [Apostol, 1974, Theorem 7.28] and [Rudin, 1964, Theorem 4.10] continuity is a sufficient condition for Riemann-integrability of a vector valued function, so $\boldsymbol{f} \in \mathcal{R}(I)$. Let $||\cdot||$ be a norm on $\mathbb{R}^N$. By lemma 3.5 and

†The implication is clearly *not* true for non-autonomous vector fields which are Lipschitz with respect to the second variable.

since $\boldsymbol{f}$ is continuous, the function $t \mapsto ||\boldsymbol{f}(t)||$ is continuous, since compositions of continuous functions are continuous ([Apostol, 1974, Theorem 4.17]). Hence, we have $||\boldsymbol{f}|| \in \mathcal{R}(I)$.

Let

$$\boldsymbol{\sigma} := \int_I \boldsymbol{f}(t)dt \quad \text{and } \sigma' := \int_I ||\boldsymbol{f}(t)||dt.$$

By definition, $\boldsymbol{f} \in \mathcal{R}(I)$ if every component of $\boldsymbol{f}$ is Riemann-integrable, $f_j \in \mathcal{R}(I)$, $j = 1,2,\ldots,N$, i.e. if σ_j denotes the jth component of $\boldsymbol{\sigma}$, we have (using a variant of [Apostol, 1974, Definition 7.1])

$$\sigma_j = \lim_{||\mathcal{P}||\to 0} \sum_{j=1}^{n} f_j(t_i^{\mathcal{P}})(x_i - x_{i-1}),$$

where $\mathcal{P} = \{x_1,x_2,\ldots,x_n\}$ such that $x_1 < x_2 < \cdots < x_n$ with $x_1 = \inf I$, $x_n = \sup I$ is a *partition* of I with the "norm" $||\mathcal{P}|| := \max_k x_k - x_{k-1}$ and $(t_1^{\mathcal{P}},t_2^{\mathcal{P}},\ldots,t_n^{\mathcal{P}})$ is any n-tuple such that $t_k^{\mathcal{P}} \in [x_k; x_{k-1}]$. Gathering the components in a vector, we can write

$$\boldsymbol{\sigma} = \lim_{||\mathcal{P}||\to 0} \sum_{j=1}^{n} \boldsymbol{f}(t_i^{\mathcal{P}})(x_i - x_{i-1}).$$

By the triangle inequality, which holds in any normed space, for any partition $\mathcal{P}$ of I, we have

$$\left\| \sum_{i=1}^{n} \boldsymbol{f}(t_i^{\mathcal{P}})(x_i - x_{i-1}) \right\| \le \sum_{i=1}^{n} ||\boldsymbol{f}(t_i^{\mathcal{P}})||(x_i - x_{i-1}). \tag{4.1}$$

In particular, as $||\mathcal{P}|| \to 0$, we see that

$$||\boldsymbol{\sigma}|| = \lim_{||\mathcal{P}||\to 0} \left\| \sum_{i=1}^{n} \boldsymbol{f}(t_i^{\mathcal{P}})(x_i - x_{i-1}) \right\| \le \lim_{||\mathcal{P}||\to 0} \sum_{i=1}^{n} ||\boldsymbol{f}(t_i^{\mathcal{P}})||(x_i - x_{i-1}) = \sigma',$$

since if this was not true, we would have that (4.1) was not true for some $\mathcal{P}$. Hence

$$\left\| \int_I \boldsymbol{f}(t)dt \right\| \le \int_I ||\boldsymbol{f}(t)||dt.$$

□

Lemma 4.1. *Let I be a closed interval and suppose $\boldsymbol{f} : I \to \mathbb{R}^N$ is continuous. Then for $a,b \in I$*

$$\left\| \int_a^b \boldsymbol{f}(t)dt \right\| \le \left| \int_a^b ||\boldsymbol{f}(t)||dt \right|.$$

Proof. By [Apostol, 1974, Definition 7.5], we have

$$\left\|\int_a^b \boldsymbol{f}(t)dt\right\| = \left\|(-1)\int_a^b \boldsymbol{f}(t)dt\right\| = \left\|\int_b^a \boldsymbol{f}(t)dt\right\|. \tag{4.2}$$

The inequality is of course true when $a = b$, so assume $a < b$. Now, by proposition 4.1

$$\left\|\int_a^b \boldsymbol{f}(t)dt\right\| \leq \int_a^b ||\boldsymbol{f}(t)||dt \leq \left|\int_a^b ||\boldsymbol{f}(t)||dt\right|.$$

Similarly assume $a > b$. Using (4.2) we get

$$\begin{aligned}\left\|\int_a^b \boldsymbol{f}(t)dt\right\| &= \left\|\int_b^a \boldsymbol{f}(t)dt\right\| \leq \int_b^a ||\boldsymbol{f}(t)||dt \\ &= -\int_a^b ||\boldsymbol{f}(t)||dt \leq \left|\int_a^b ||\boldsymbol{f}(t)||dt\right|.\end{aligned}$$

□

Notation. Let $f : X \to Y$ be a function. We will use the notation $f|_Z$ for the restriction of f to the set $Z \subseteq X$, i.e.

$$f|_Z(x) := f(x), \quad \forall\, x \in Z.$$

The following lemma concerns the properties of the mapping of (continuous) curves defined by the right hand side of the integral equation (2.5); no specific references to the IVP are needed yet.

Lemma 4.2. *Suppose $\mathcal{O} = I \times S \subseteq \mathbb{R} \times \mathbb{R}^N$ and let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ be a continuous vector field. Assume that $(t_0,\boldsymbol{x}_0) \in \mathcal{O}$ and define the mapping $T : \mathcal{C}(I,S) \to \mathcal{C}(I,S)$ by*

$$T(\boldsymbol{x})(t) := \boldsymbol{x}_0 + \int_{t_0}^t \boldsymbol{X}(s,\boldsymbol{x}(s))ds.$$

Then there exists a restriction $T|_C$ to a set $C \subset \mathcal{C}(I,S)$ such that

$$T|_C : C \to C.$$

Proof. Note that the defined mapping is just the right hand side of (2.5) and therefore well-defined. In the following, let $||\cdot||$ be any fixed norm on $\mathbb{R}^N$.

Restrict $\boldsymbol{X}$ to a compact neighbourhood around $(t_0,\boldsymbol{x}_0)$, say

$$N_{(t_0,\boldsymbol{x}_0)} = [t_0 - a, t_0 + a] \times \overline{B(\boldsymbol{x}_0, r)}$$

By assumption, the vector field $\boldsymbol{X}$ is continuous, and since it is defined on a cartesian product of to compact sets, it is defined on a compact set, [Gleason, 1966, Corollary 14-8.9], so by [Apostol, 1974, Theorem 4.27], $\boldsymbol{X}$ is bounded.

$$\exists\, M \geq 0 : ||\boldsymbol{X}(t,\boldsymbol{x})|| \leq M, \quad \forall\, (t,\boldsymbol{x}) \in N_{(t_0,\boldsymbol{x}_0)}. \tag{4.3}$$

Consider $T|_C$ with $C := \mathcal{C}([t_0-\varepsilon; t_0+\varepsilon], \overline{B(\boldsymbol{x}_0,r)})$ and $\varepsilon > 0$ chosen such that

$$\varepsilon < a, \quad M\varepsilon < r.$$

We now show that with this choice of ε, we have $T|_C : C \to C$. Now by lemma 4.1

$$\begin{aligned} ||T(\boldsymbol{x})(t) - \boldsymbol{x}_0|| &= \left\| \int_{t_0}^{t} \boldsymbol{X}(s,\boldsymbol{x}(s))ds \right\| \\ &\leq \left| \int_{t_0}^{t} ||\boldsymbol{X}(s,\boldsymbol{x}(s))||ds \right| \end{aligned}$$

for any choice of $t \in [t_0 - \varepsilon; t_0 + \varepsilon]$. Because of (4.3), and since integration is order preserving, cf. [Apostol, 1974, Theorem 7.20], we have

$$\begin{aligned} ||T(\boldsymbol{x})(t) - \boldsymbol{x}_0|| &\leq \left| \int_{t_0}^{t} ||\boldsymbol{X}(s,\boldsymbol{x}(s))||ds \right| \\ &\leq |M(t-t_0)| = M|t-t_0| \leq M\varepsilon < r. \end{aligned}$$

Therefore, $T(\boldsymbol{x})(t) \in B(\boldsymbol{x}_0,r)$ and we see that $T|_C$ maps C into itself. □

According to lemma 4.2, we can restrict the mapping T to a certain subset of the space of continuous functions and deduce that this subset is mapped into itself by T. Applying theorem 3.4 shows that when equipped with the sup norm, this set of functions is a closed subset of a Banach space and hence a complete metric space; this is an important observation for the next theorem.

Theorem 4.1 (Local existence and uniqueness). *Suppose $\mathcal{O} = I \times S \subseteq \mathbb{R} \times \mathbb{R}^N$ and let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ be a continuous vector field which satisfies a local Lipschitz condition in the second variable. Then the IVP with initial condition $(t_0,\boldsymbol{x}_0) \in \mathcal{O}$ generated by $\boldsymbol{X}$ has a unique solution $\boldsymbol{\alpha} : I \to \mathbb{R}^N$ defined on some sufficiently small open interval of times containing t_0.*

Proof. Let $||\cdot||$ be a norm on $\mathbb{R}^N$. Choose a compact neighbourhood

$$N_{(t_0,\boldsymbol{x}_0)} := [t_0 - a, t_0 + a] \times \overline{B(\boldsymbol{x}_0, r)} \subseteq \mathcal{O},$$

such that $\boldsymbol{X}$ satisfies a Lipschitz condition in the second variable on $N_{(t_0,\boldsymbol{x}_0)}$; fix a Lipschitz constant L_0. We employ the integral equation of the IVP, cf. proposition 2.2, and define T as in lemma 4.2. Since we have restricted $\boldsymbol{X}$ to a compact neighbourhood, there exists $\varepsilon > 0$ such that the restriction of T to the set C given by

$$C := \mathcal{C}([t_0 - \varepsilon; t_0 + \varepsilon], \overline{B(\boldsymbol{x}_0, r)}),$$

gives $T|_C : C \to C$. Note that any set of the above form, when endowed with the metric d_∞ induced by the sup norm, is a complete metric space, see proposition 3.5 (since it is a closed subset of the Banach space of continuous functions $(\mathcal{C}([t_0 - \varepsilon; t_0 + \varepsilon], \mathbb{R}^N), ||\cdot||_\infty)$) Further restriction on T will make it possible to show that T defines a contraction on a such metric space of continuous functions with respect to the metric induced by the sup norm.

Choose $\tilde{\varepsilon}$ such that

$$\tilde{\varepsilon} L_0 < 1, \quad \tilde{\varepsilon} \leq \varepsilon,$$

and define

$$C_{\tilde{\varepsilon}} := \mathcal{C}([t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}], \overline{B(\boldsymbol{x}_0, r)}).$$

Assume $\boldsymbol{\beta}, \boldsymbol{\gamma} \in C_{\tilde{\varepsilon}}$. Now, by lemma 4.1

$$\begin{aligned}
||T(\boldsymbol{\beta})(t) - T(\boldsymbol{\gamma})(t)|| &= \left|\left| \int_{t_0}^{t} \{\boldsymbol{X}(s, \boldsymbol{\beta}(s)) - \boldsymbol{X}(s, \boldsymbol{\gamma}(s))\} ds \right|\right| \\
&\leq \left| \int_{t_0}^{t} ||\boldsymbol{X}(s, \boldsymbol{\beta}(s)) - \boldsymbol{X}(s, \boldsymbol{\gamma}(s))|| ds \right|.
\end{aligned}$$

Since, by assumption, $\boldsymbol{X}$ satisfies a Lipschitz condition in the second variable with Lipschitz constant L_0 we have

$$\begin{aligned}
||T(\boldsymbol{\beta})(t) - T(\boldsymbol{\gamma})(t)|| &\leq \left| \int_{t_0}^{t} ||\boldsymbol{X}(s, \boldsymbol{\beta}(s)) - \boldsymbol{X}(s, \boldsymbol{\gamma}(s))|| ds \right| \\
&\leq \left| L_0 \int_{t_0}^{t} ||\boldsymbol{\beta}(s) - \boldsymbol{\gamma}(s)|| ds \right| \\
&\leq L_0 \left| \int_{t_0}^{t} ||\boldsymbol{\beta} - \boldsymbol{\gamma}||_\infty ds \right| \\
&\leq L_0 |t - t_0| \, ||\boldsymbol{\beta} - \boldsymbol{\gamma}||_\infty \leq L_0 \tilde{\varepsilon} ||\boldsymbol{\beta} - \boldsymbol{\gamma}||_\infty,
\end{aligned}$$

where the third inequality comes from [Apostol, 1974, Theorem 7.20], since by definition $||\boldsymbol{\beta}(t) - \boldsymbol{\gamma}(t)|| \leq ||\boldsymbol{\beta} - \boldsymbol{\gamma}||_\infty$ for all t. Therefore

$$||T(\boldsymbol{\beta}) - T(\boldsymbol{\gamma})||_\infty \leq L_0\tilde{\varepsilon}||\boldsymbol{\beta} - \boldsymbol{\gamma}||_\infty.$$

Now $\tilde{\varepsilon}$ was chosen to satisfy $L_0\tilde{\varepsilon} < 1$. According to definition 3.6, the operator T is a contraction map on the complete metric space $(C_{\tilde{\varepsilon}}, d_\infty)$, and by Banach's fixed point theorem, theorem 3.1, T has a unique fixed point $\tilde{\boldsymbol{\alpha}} : [t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}] \to \overline{B(\boldsymbol{x}_0, r)}$. Defining $\boldsymbol{\alpha} := \tilde{\boldsymbol{\alpha}}|_{]t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}[}$, we obtain a curve differentiable on its entire interval of definition which satisfies the integral equation of the IVP and therefore satisfies the IVP, by proposition 2.2. □

Remark. Although the above proof states that there exists a unique solution defined as the fixed point of the mapping T on the complete metric space $C_{\tilde{\varepsilon}}$ w.r.t. the metric induced by the sup norm, there is a technical complication. The complete metric spaces $C_{\tilde{\varepsilon}}$ for different $\tilde{\varepsilon}$ comprises a one-parameter family dependent on $\tilde{\varepsilon}$; and we cannot a priori be sure that various choices of $\tilde{\varepsilon}$ yields the same fixed point.

However, we can still ensure uniqueness. Suppose $\boldsymbol{\alpha} : [t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}] \to \mathbb{R}^N$ is the solution as generated above, and that $\boldsymbol{\beta} : J \to \mathbb{R}^N$ is another solution of the IVP.

If $[t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}] \subseteq J$ then the restriction $\boldsymbol{\beta}|_{[t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}]}$ is still a solution, i.e. it is a fixed point of T, and because of the choice of $\tilde{\varepsilon}$ we see that

$$\big|\big|\boldsymbol{\beta}|_{[t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}]} - \boldsymbol{x}_0\big|\big| = \big|\big|T(\boldsymbol{\beta}|_{[t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}]}) - \boldsymbol{x}_0\big|\big| < r,$$

as in lemma 4.2, so $\boldsymbol{\beta}|_{[t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}]}$ is an element in the same space as $\boldsymbol{\alpha}$. Now by uniqueness of $\boldsymbol{\alpha}$ we have

$$\boldsymbol{\beta}|_{[t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}]} = \boldsymbol{\alpha}.$$

Now if $J \subset [t_0 - \tilde{\varepsilon}; t_0 + \tilde{\varepsilon}]$ we can restrict $\boldsymbol{\alpha}$ further and again obtain uniqueness. This means that every other solution will coincide with the one found in theorem 4.1 on the intersection of their intervals.

4.3 Global Existence and Uniqueness

The results of the previous sections stated that an IVP generated by a (non-autonomous) vector field $\boldsymbol{X}$ always has a unique solution on some sufficiently small interval of times, assuming that $\boldsymbol{X}$ is continuous and satisfies a local

Lipschitz condition in the second variable. This section is devoted to improving this result — we will prove that any locally unique solution can be extended to a unique solution defined on the largest possible interval of times.

More importantly, we will show that if $\boldsymbol{X}$ in addition is bounded and satisfies a local Lipschitz condition in the second variable, then the IVP generated by $\boldsymbol{X}$ has solutions defined on the whole $\mathbb{R}$. These demands are fulfilled by the vector field generating our IVP, as we will see in section 4.3.1.

Definition 4.2 (Maximal solution). *A solution $\boldsymbol{\alpha} : I \to \mathbb{R}^N$ to an IVP is called the maximal solution of the IVP if, for any other solution $\boldsymbol{\beta} : J \to \mathbb{R}^N$, we have*

$$J \subseteq I, \quad \text{and } \boldsymbol{\alpha}(t) = \boldsymbol{\beta}(t), \quad \forall\, t \in I \cap J.$$

The interval I is called the maximal interval of existence.

We now present the method of continuation of solutions. This will provide us with a constructive proof of existence and uniqueness of maximal solutions.

Proposition 4.2 (Continuation of solution). *Suppose $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ satisfies a local Lipschitz condition in the second variable. Then given a unique solution $\boldsymbol{\alpha} : I \to \mathbb{R}^N$ to the IVP generated by $\boldsymbol{X}$, suppose one or both of the following are fulfilled*

(i) $\boldsymbol{\varphi}_+ := \lim_{t \uparrow \sup I} \boldsymbol{\alpha}(t)$ *exists and* $(\sup I, \boldsymbol{\varphi}_+) \in \mathcal{O}$.

(ii) $\boldsymbol{\varphi}_- := \lim_{t \downarrow \inf I} \boldsymbol{\alpha}(t)$ *exists and* $(\inf I, \boldsymbol{\varphi}_-) \in \mathcal{O}$.

Then $\boldsymbol{\alpha}$ can be continued to a solution $\boldsymbol{\beta} : J \to \mathbb{R}^N$ with $I \subseteq J$ if one or both of the above are fulfilled

Proof. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^N$ be a unique solution defined on some open interval I containing t_0. By theorem 4.1, such a solution exists.

Let $a := \sup I$. Suppose now, that the limit

$$\boldsymbol{\varphi}_+ := \lim_{t \uparrow \sup I} \boldsymbol{\alpha}(t),$$

exists and that $(a, \boldsymbol{\varphi}_+) \in \mathcal{O}$. In this case, we construct the following IVP

$$\left.\begin{aligned} \boldsymbol{x}' &= \boldsymbol{X}(t, \boldsymbol{x}) \\ \boldsymbol{x}(a) &= \boldsymbol{\varphi}_+. \end{aligned}\right\} \tag{4.4}$$

Since $\boldsymbol{X}$ satisfies a local Lipschitz condition in x, we conclude that there exists $\varepsilon > 0$ and a unique solution $\boldsymbol{\beta} : [a; a+\varepsilon[\to \mathbb{R}^N$. Define the following curve

$$\boldsymbol{\gamma}(t) := \begin{cases} \boldsymbol{\alpha}(t) & \text{if } t \in [t_0; a[\\ \boldsymbol{\beta}(t) & \text{if } t \in [a; a+\varepsilon[. \end{cases}$$

Clearly, since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are continuous, $\boldsymbol{\gamma}$ is also continuous. We claim that $\boldsymbol{\gamma}$ satisfies the original IVP. We have that

$$\boldsymbol{\gamma}(t) = \boldsymbol{\beta}(a) + \int_a^t \boldsymbol{X}(s,\boldsymbol{\gamma}(s))ds, \qquad \text{for } t \in [a; a+\varepsilon[\tag{4.5}$$

but

$$\boldsymbol{\beta}(a) = \boldsymbol{x}_0 + \int_{t_0}^a \boldsymbol{X}(s,\boldsymbol{\gamma}(s))ds. \tag{4.6}$$

Inserting (4.6) into (4.5) we obtain, using the properties of the Riemann integral [Apostol, 1974, Theorem 7.4]

$$\begin{aligned} \boldsymbol{\gamma}(t) &= \boldsymbol{x}_0 + \int_{t_0}^a \boldsymbol{X}(s,\boldsymbol{\gamma}(s))ds + \int_a^t \boldsymbol{X}(s,\boldsymbol{\gamma}(s))ds \\ &= \boldsymbol{x}_0 + \int_{t_0}^t \boldsymbol{X}(s,\boldsymbol{\gamma}(s))ds, \qquad \text{for } t \in [a; a+\varepsilon[\end{aligned}$$

Obviously, we also have

$$\boldsymbol{\gamma}(t) = \boldsymbol{x}_0 + \int_{t_0}^t \boldsymbol{X}(s,\boldsymbol{\gamma}(s))ds, \qquad \text{for } t \in [t_0; a+\varepsilon[.$$

This implies that $\boldsymbol{\gamma}$ solves the original IVP and we have now constructed a solution defined on a larger interval of times. Furthermore, this solution is unique, since by assumption $\boldsymbol{\alpha}$ is the unique solution to the original IVP on $[t_0; a[$ and $\boldsymbol{\beta}$ is also uniquely determined on the interval $[a; a+\varepsilon[$. In a similar way we can extend the solution to the left provided that

$$\boldsymbol{\varphi}_- := \lim_{t \downarrow \inf I} \boldsymbol{\alpha}(t),$$

exists and $(\inf I, \boldsymbol{\varphi}_-) \in \mathcal{O}$. □

We observe, that if the limits $\boldsymbol{\varphi}_+$ and $\boldsymbol{\varphi}_-$ do not exist, then the solution approaches plus or minus infinity as we approach the endpoints of the interval on which the solution is defined. This makes it impossible to continue the solution. Similarly, if we approach the boundary of $\mathcal{O}$ when approaching the endpoints, the solution cannot be continuated any further. Figure 4.1 shows

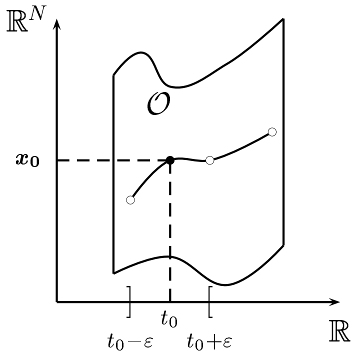

Figure 4.1: *The principle of continuation of a solution. Given an IVP generated by a continuous, locally Lipschitz vector field, we can always determine a unique local solution in a small interval containing the initial time. However, if the limits of the solution at one or both of the endpoints exist, we can define a new IVP; and continue the solution.*

the principle of continuation of solutions These observations make the idea of a unique maximal solution more intuitive — we can always extend a solution until we (if possible) reach the boundary of the range or the solution diverges to infinity.

Theorem 4.2 (Unique maximal solution). *Suppose $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ satisfies a local Lipschitz condition in the second variable. Then there exists a unique maximal solution to the IVP generated by $\boldsymbol{X}$.*

Proof. Given a unique solution $\boldsymbol{\alpha} : I \to \mathbb{R}^N$ to the IVP (which by local existence and uniqueness, theorem 4.1, exists), we can sequentially extend this solution to solutions defined on $I_1, \dots, I_n$ satisfying

$$I \subseteq I_1 \subseteq \cdots \subseteq I_n.$$

If for some I_j, $j = 1,2,\dots,n-1$ the solution cannot be extended any further, we set $I_{j+k} := I_j$ for $k = 1,\dots,n-j$.

By proposition 4.2, for each I_j, there exists a unique solution $\boldsymbol{\alpha}_j : I_j \to \mathbb{R}^N$, $j = 1,\dots n$. Now, let

$$b := \lim_{n\to\infty} (\sup I_n) \text{ and } a := \lim_{n\to\infty} (\inf I_n).$$

Note that it is possible that $b = \infty$ and/or $a = -\infty$. Let $I_{(t_0,\boldsymbol{x}_0)} :=]a; b[$.

Now if $\boldsymbol{\beta} : J \to \mathbb{R}^N$ also solves the IVP then since J by definition 2.3 is an open interval, and since $\boldsymbol{\beta}(t_0)$ must be defined and equal $\boldsymbol{x}_0$, we conclude that $J \subseteq I_{(t_0,\boldsymbol{x}_0)}$. But because of the discussion following theorem 4.1 we see that $\boldsymbol{\alpha}(t) = \boldsymbol{\beta}(t)$ for all $t \in J$. □

Notation. For the rest of the report, the notation $\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)} : I_{(t_0,\boldsymbol{x}_0)} \to \mathbb{R}^N$ will *always* mean the maximal solution satisfying $\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t_0) = \boldsymbol{x}_0$.

Theorem 4.3 (Global existence). *Let $\mathcal{O} = \mathbb{R} \times \mathbb{R}^N$, and suppose that $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ is bounded, continuous, and satisfies a local Lipschitz condition in the second variable. Then the IVP generated by $\boldsymbol{X}$ has a maximal solution defined on the whole $\mathbb{R}$*

Proof. Suppose $\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)} : I_{(t_0,\boldsymbol{x}_0)} \to \mathbb{R}^N$ is the maximal solution of the initial value problem — by theorem 4.2, the maximal solution always exists. We wish to show that $I_{(t_0,\boldsymbol{x}_0)} = \mathbb{R}$.

Now assume that $\inf I_{(t_0,\boldsymbol{x}_0)} = a > -\infty$. Since $I_{(t_0,\boldsymbol{x}_0)}$ is an open interval a is an accumulation point of $I_{(t_0,\boldsymbol{x}_0)}$ and by [Apostol, 1974, Theorem 3.17] every n-ball around a contains infinitely many points of $I_{(t_0,\boldsymbol{x}_0)}$. This, by [Apostol, 1974, Theorem 4.4], means that there exists a sequence $\{t_n\}_{n\geq 1}$ in $I_{(t_0,\boldsymbol{x}_0)}$ converging to a.

Now this sequence is a Cauchy sequence. The strategy is now to prove that $\{\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t_n)\}_{n\geq 1}$ is also a Cauchy sequence. Without loss of generality, we can assume $t_n < t_m$

$$\begin{aligned}
||\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t_m) - \boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t_n)|| &= \left\| \int_{t_n}^{t_m} \boldsymbol{X}(\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(s))ds \right\| \\
&\leq \int_{t_n}^{t_m} \left\| \boldsymbol{X}(\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(s)) \right\| ds \\
&\leq (t_m - t_n) \sup_{t_n \leq t \leq t_m} ||\boldsymbol{X}(\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t))||.
\end{aligned}$$

By assumption $\boldsymbol{X}$ is bounded on $\mathcal{O}$ so $||\boldsymbol{X}(\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t)|| \leq M$ for some constant $M \geq 0$.

Pick $\varepsilon > 0$. Now, since $\{t_n\}_{n\geq 1}$ is Cauchy, choosing

$$\varepsilon' = \frac{\varepsilon}{M+1},$$

there exists $\tilde{N}(\varepsilon')$ such that for $n,m \geq \tilde{N}(\varepsilon')$ we have $|t_m - t_n| < \varepsilon'$. Therefore, choosing $n,m \geq \tilde{N}(\varepsilon')$ we obtain

$$\begin{aligned}
||\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t_m) - \boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t_n)|| &< M|t_m - t_n| \\
&< M\frac{\varepsilon}{M+1} \\
&< \varepsilon.
\end{aligned}$$

This proves that $\{\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t_n)\}_{n\geq 1}$ is a Cauchy sequence and since every Cauchy sequence converges in $\mathbb{R}^N$ [Apostol, 1974, theorem 4.8], the sequence converges, i.e. $\lim_{n\to\infty} \boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t_n)$ exists.

But then proposition 4.2 implies that we can construct an even larger interval on which a (unique) solution is defined. This is a contradiction and we conclude $a = -\infty$. Similarly we could assume $\sup I_{(t_0,\boldsymbol{x}_0)} = b < \infty$ and let a $\{t_n\}_{n\geq 1}$ converge to b, which is possible by the same arguments as above, and we would again obtain a contradiction. So $I_{(t_0,\boldsymbol{x}_0)} = \mathbb{R}$. □

4.3.1 Lipschitz Condition and Boundedness for $\boldsymbol{X}(x,v)$

We will now show that the vector field of our system, $\boldsymbol{X}(x,v) = (v,F(x))$ with F given by (1.2) is bounded and satisfies a global Lipschitz condition. According to theorem 4.3, this implies that there exists a unique solution defined on the whole $\mathbb{R}_0^+$ regardless of initial conditions (in fact the solution is defined on $\mathbb{R}$ but we restrict ourselves to non-negative values).

Proposition 4.3. *Suppose $\frac{1}{2}v^2 + V(x) = E < \infty$. Then the vector field $\boldsymbol{X}(x,v)$ given by (1.2) is bounded.*

Proof. For $\boldsymbol{X}(x,v)$ to be bounded, we demand that there exists $M \geq 0$ such that

$$||\boldsymbol{X}(x,v)|| := |v| + |F(x)| \leq M.$$

We have $-1 \leq F(x) \leq 1 \Rightarrow |F(x)| \leq 1$ for all $x \in \mathbb{R}$. Regarding v, we assume that the energy E is finite so

$$\frac{1}{2}v^2 + V(x) = E \Rightarrow v = \pm\sqrt{2E - 2V(x)}.$$

But $-3/4 \leq V(x) \leq 0$ for all $x \in \mathbb{R}$, so v must also be bounded. Therefore there exists $M \geq 0$ such that

$$||\boldsymbol{X}(x,v)|| \leq M.$$

□

In the proof that the vector field satisfies a (global) Lipschitz condition the following lemma states that it is sufficient to determine a Lipschitz condition for the force $F(x)$. For the following, we recall (from the remark following definition 4.1) that Lipschitz continuity is independent of the choice of norm (on finite-dimensional spaces).

Lemma 4.3. *Suppose the function* $F : \mathbb{R} \to \mathbb{R}$ *as given by* (1.2) *is globally Lipschitz. Then the vector field* $\boldsymbol{X} : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $\boldsymbol{X}(x,v) = (v,F(x))$ *is globally Lipschitz.*

Proof. Assume F is globally Lipschitz. This means, stated in terms of the ℓ_1-norm

$$\exists\, K \geq 0 \,:\, x,y \in \mathbb{R} \Rightarrow |F(x) - F(y)| \leq K|x - y|.$$

Adding $|z - w|$ with $z,w \in \mathbb{R}$ on both sides gives

$$|F(x) - F(y)| + |z - w| \leq K|x - y| + |z - w|.$$

Take $L = \max\{K,1\}$. Then

$$\begin{aligned} |F(x) - F(y)| + |z - w| \leq L|x - y| + L|z - w| \Leftrightarrow \\ ||\boldsymbol{X}(x,z) - \boldsymbol{X}(y,w)||_1 \leq L||(x,z) - (y,w)||_1. \end{aligned}$$

□

The above lemma shows that for our particular system, it is sufficient to prove that the function F satisfies a global Lipschitz condition. In the general case of a C^1 function $f : \mathbb{R} \to \mathbb{R}$ with bounded derivative, Lipschitz continuity follows directly from the mean value theorem, [Apostol, 1974, Theorem 5.11], which states that given any two points x,y (assume $x < y$), there exists $x < \gamma < y$ such that

$$f(x) - f(y) = f'(\gamma)(x - y) \Rightarrow |f(x) - f(y)| \leq K|x - y|,$$

see also the discussion on p. 37. Our particular function $F(x)$ is unfortunately not this well-behaved, since it is only piecewise C^1 on $\mathbb{R}$, cf. p. 2.

However, an exhaustive investigation of all possibilities will show that the function indeed satisfies a global Lipschitz condition.

Theorem 4.4 (Lipschitz condition for our system). *The function* F*, as defined by* (1.2)*, satisfies a global Lipschitz condition.*

Proof. We will investigate all possible configurations for x and y. A quick estimate tells us that there exists 15 possible configurations. Five of these corresponds to the case where x,y belongs to the same segment; since each function in the piecewise function is clearly C^1, we can apply the mean value theorem, [Apostol, 1974, Theorem 5.11], and conclude that a Lipschitz condition is satisfied.

In the remaining cases, we will distinguish between two different cases

(i) x and y belong to neighbouring segments;

(ii) x and y do not belong to neighbouring segments (and are not in the same segment).

We claim that Lipschitz condition for (i) implies that a Lipschitz condition for (ii) is satisfied. To see this, suppose x and y belong to segments that are *not* neighbours, $x \in I_1$ and $y \in I_2$ and that F satisfies a Lipschitz condition for all neighbouring segments. Then there are at most two segments between x and y, say J_1 and J_2. Now, suppose $a \in J_1$ and $b \in J_2$. By assumption there exists K_1,K_2,$K_3 \geq 0$ such that

$$\begin{aligned}|F(x) - F(a)| &\leq K_1|x - a|,\\ |F(b) - F(y)| &\leq K_2|b - y|,\\ |F(a) - F(b)| &\leq K_3|a - b|.\end{aligned}$$

Hence

$$\begin{aligned}|F(x) - F(y)| &\leq |F(x) - F(a)| + |F(b) - F(y)| + |F(a) - F(b)|\\ &\leq K_1|x - a| + K_2|b - y| + K_3|a - b|\\ &\leq (K_1 + K_2 + K_3)|x - y|,\end{aligned}$$

because $|x-y|$ is greater than either $|x-a|$, $|b-y|$, and $|a-b|$ by assumption.

Therefore we only need to show that Lipschitz condition is satisfied for the four different cases of (i). Denote the segments determined by the piecewise function F by

$$I_1 :=] - \infty; -2],\ I_2 :=] - 2; -1],\ I_3 :=] - 1; 1],\ I_4 :=]1; 2],\ I_5 :=]2; \infty[.$$

We now have the four cases, each of which is proved by manipulating the expressions using the triangle inequality and estimates of relative

Case I: Let $x \in I_1$ and $y \in I_2$

$$\begin{aligned}|F(x) - F(y)| &= | - y - 2|\\ &= | - y - 2 + x - x|; \quad \text{add } x - x.\\ &\leq |x - y| + | - x - 2|; \quad \text{triangle ineq.}\\ &\leq 2|x - y|.\end{aligned}$$

In the last inequality, we use the estimate $| - x - 2| < |x - y|$ since we assume $y \in I_2$.

Case II: Assume $x \in I_2$, $y \in I_3$.

$$\begin{aligned}
|F(x)-F(y)| &\le |F(x)-F(-1)| + |F(-1)-F(y)|; \quad \text{triangle ineq.}\\
&\le |x+1| + |1+y^3|; \quad \text{add x-x.}\\
&\le |x+1| + |x+1| + |y^3-x|\\
&\le |x+1| + |x+1| + |y-x|\\
&\le 3|x-y|.
\end{aligned}$$

Note that $|y^3 - x| \le |y-x|$ since $y \in I_3$. Also, $|x+1| \le |x-y|$, by the same reasoning as in case I.

Case III: $x \in I_3$, $y \in I_4$.

$$\begin{aligned}
|F(x)-F(y)| &\le |F(x)-F(1)| + |F(1)-F(y)|; \quad \text{triangle ineq.}\\
&\le 3|x-1| + |1-y|\\
&\le 3|x-1| + |1-x| + |x-y|; \quad \text{add x-x.}\\
&\le 4|x-1| + |x-y|\\
&\le 5|x-y|; \quad \text{as in case I and II.}
\end{aligned}$$

Case IV: $x \in I_4$, $y \in I_5$.

$$\begin{aligned}
|F(x)-F(y)| &\le |F(x)-F(2)| + |F(2)-F(y)|; \quad \text{triangle ineq.}\\
&\le |x-2|\\
&\le |x-y|; \quad \text{as in case I, II and III.}
\end{aligned}$$

Since F satisfies a Lipschitz condition in all neighbouring segments, it satisfies a Lipschitz condition on $\mathbb{R}$. This completes the proof. $\square$

The above proof suffices for our purpose — however, for the sake of completeness we present the following much more general theorem for functions of one variable. It states that if a function is piecewise continuously differentiable (which implies that the set of discontinuities for the derivative is finite) and the derivative bounded, then the function satisfies a global Lipschitz condition. This is a consequence of Lebesgue's criterion for Riemann integrability, [Apostol, 1974, Theorem 7.48]. The following proof is based on [Perko, 1991, p. 71].

Theorem 4.5 (Sufficient condition). *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a piecewise differentiable function having a piecewise continuous, bounded derivative. Then f satisfies a global Lipschitz condition and*

$$L = \sup_{x\in\mathbb{R}} |f'(x)|.$$

Proof. By assumption, $f'(x)$ is bounded so define

$$L := \sup_{x\in\mathbb{R}} |f'(x)| < \infty.$$

Let $x,y \in \mathbb{R}$ be given and define $u = y - x$. Since the interval $I = [x; y]$ is merely a closed ball in $\mathbb{R}$, it is convex and therefore

$$w := x + su \in I, \quad 0 \le s \le 1.$$

Now consider the function $s \mapsto f(x + su)$ and denote it by F.

Clearly, $f(y) - f(x) = F(1) - F(0)$. Applying the chain rule, we get

$$\frac{dF}{ds} = \frac{dF}{dw}\frac{dw}{ds} = F'(x + su)u.$$

Note that F may be undefined in some points on its domain. However, we have $F'(x + su) \in \mathcal{R}([0; 1])$, according to Lebesgue's criterion for Riemann-integrability, [Apostol, 1974, 7.48]; since the number of points in which F' is discontinuous is finite (and thus of measure zero) and $F'(x+su)$ by assumption is bounded. In fact we have

$$f(y) - f(x) = F(1) - F(0) = \int_0^1 \frac{d}{ds}\left(F(x + su)\right)\, ds = \int_0^1 F'(x + su)u\, ds.$$

Applying proposition 4.1 we see that

$$\begin{aligned} |f(y) - f(x)| &\le \int_0^1 |F'(x + su)u| ds \\ &\le \int_0^1 |u| \sup_{s\in[0;1]} |F'(x + su)|\, ds \\ &\le |u|L = L|y - x|. \end{aligned}$$

Since x,y were arbitrarily chosen, f satisfies a global Lipschitz condition. □

Proposition 4.4 (Global existence and uniqueness for our system). *The IVP generated by* (2.3) *has a unique solution defined for all $t \in \mathbb{R}_0^+$.*

Proof. Since $\boldsymbol{X}$ is bounded and satisfies a global Lipschitz condition, we can apply theorem 4.3 in the restricted case where we assume $t \ge 0$. □

As noted in theorem 4.3 the solutions could actually be defined for $t \in \mathbb{R}$, but from a physical point of view the negative times are uninteresting.

CHAPTER 5

Solutions in the Linear Regions

In section 2.1.1, we mentioned that for the regions $x(t) \leq -1$ and $1 < x(t)$ our system (2.3) is linear and in fact reduces to a second-order equation with constant coefficients.

The purpose of this chapter is to devise how this solution can be calculated using general methods applicable for any constant coefficient first-order system. In the study of the dynamics of the system, obtaining explicit solutions are not necessarily worth the effort — however, they will help us in the analysis of stability which follows in section 6.2.

Our approach is based on the general concept of the exponential function of continuous, linear operators mappings acting on a Banach space — we will be interested in the case where the Banach space is simply $\mathbb{R}^N$ and the operators therefore $N \times N$ matrices. This approach is theoretically the most concise, although it is difficult to apply directly for more complicated systems.

5.1 The Operator Exponential

We recall that a linear mapping between (real) vector spaces V,W is a mapping $T : V \to W$ such that for v_1,$v_2 \in V$ we have

$$T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2), \quad c_1,c_2 \in \mathbb{R}.$$

Definition 5.1. *Let V and W be vector spaces. The set of all linear, continuous mappings $T : V \to W$ will be denoted as $\mathcal{L}(V,W)$.*

It is shown in [Axler, 1997, p. 40] that the set of linear mappings from V to W is a vector space with the usual addition and scalar multiplication of mappings. Clearly $\mathcal{L}(V,W)$ is a subspace of this space, since continuity is preserved under scalar multiplication and addition of continuous mappings. In fact, $\mathcal{L}(V,W)$ is a subspace of $\mathcal{C}(V,W)$ since a sum of linear mappings is again a linear mapping.

We will now extend this to a normed vector space by introducing the *operator norm.*

Definition 5.2 (Operator norm). *Let $T \in \mathcal{L}(V,W)$, with $(V,||\cdot||_v)$ and $(W,||\cdot||_w)$ normed spaces. The function $||\cdot|| : \mathcal{L}(V,W) \to \mathbb{R}$ defined as*

$$||T|| := \sup\{||Tx||_w \mid ||x||_v \leq 1\}, \quad x \in V,$$

is called the operator norm.

Proposition 5.1. *The operator norm is a norm on $\mathcal{L}(V,W)$.*

Proof. Let $T \in \mathcal{L}(V,W)$. We must verify that the properties for the norm listed in definition 3.7 hold. Since $||x||_v \geq 0$ for any $x \in V$, it follows that

$$||T|| \geq 0.$$

Now assume that $||T|| = 0$. This means that $||Tx||_w = 0$ for all $x \in \overline{B(0,1)}$. Given any vector $x \in V$, we can write $||Tx||_w = ||x||_v \, ||T((1/||x||_v)x)||_w$ which is equivalent to saying null $T = V$. But then $T = 0$. Similarly if $T = 0$ then $||Tx||_w = 0$ for all $x \in V$ and therefore $||T|| = 0$. Consequently $||T|| = 0$ iff $T = 0$.

To verify homogeneity, let $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} ||\lambda T|| &= \sup\{||\lambda Tx||_w \mid ||x||_v \leq 1\} \\ &= \sup\{|\lambda| \, ||Tx||_w \mid ||x||_v \leq 1\} \\ &= |\lambda| \sup\{||Tx||_w \mid ||x||_v \leq 1\} \\ &= |\lambda| \, ||T||, \end{aligned}$$

by using proposition 3.3 on scalar multiplication of the supremum.

For the triangle inequality, let $S \in \mathcal{L}(V,W)$. Then

$$\begin{aligned} ||T + S|| &\leq \sup\{||(T+S)x||_w \mid ||x||_v \leq 1\} \\ &= \sup\{||Tx + Sx||_w \mid ||x||_v \leq 1\} \\ &\leq \sup\{||Tx||_w + ||Sx||_w \mid ||x||_v \leq 1\} \\ &\leq \sup\{||Tx||_w \mid ||x||_v \leq 1\} + \sup\{||Sx||_w \mid ||x||_v \leq 1\} \\ &= ||T|| + ||S||, \end{aligned}$$

where we have used that

$$||Tx||_w \leq \sup\{||Tx||_w\} \text{ and } ||Sx||_w \leq \sup\{||Sx||_w\} \Rightarrow$$
$$||Tx||_w + ||Sx||_w \leq \sup\{||Tx||_w\} + \sup\{||Sx||_w\},$$

which for $||x||_v \leq 1$ gives the desired inequality. □

Hence $(\mathcal{L}(V,W), ||\cdot||)$ is a normed space; this makes it possible to consider convergence, limits, etc. using the properties of the underlying metric space, cf. the discussion on p. 20.

The next proposition gives some additional properties of the operator norm; a proof can be found in [Perko, 1991, Lemma, p. 11].

Notation. Let V be a vector space. We define $\mathcal{L}(V) := \mathcal{L}(V,V)$.

Proposition 5.2. *Let $T, S \in \mathcal{L}(V)$ and $x \in V$. Then*

$$||Tx||_w \leq ||T||\,||x||_v, \text{ and } ||TS|| \leq ||T||\,||S||.$$

Using induction it also follows that $||T^n|| \leq ||T||^n$. This is easy to see; for the case $n = 2$ the statement is certainly true. For $n > 1$, we can write the product $T^{n+1} = T(T^n)$ so $||T^{n+1}|| = ||T(T^n)|| \leq ||T||\,||T^n||$.

The next theorem is also presented without proof. A proof is given in [Berg, 1997, Sætning 5.12]; this is in many ways similar to the proofs for theorem 3.3 and 3.4.

Theorem 5.1. *Suppose $(V,||\cdot||_v)$ is a normed space and $(W,||\cdot||_w)$ is a Banach space. Then $(\mathcal{L}(V,W),||\cdot||)$ is also a Banach space.*

The next proposition uses the completeness result, and allows us to define the matrix exponential.

Proposition 5.3. *Let V be a Banach space and assume $T \in \mathcal{L}(V)$. The series*

$$\exp(Tt) := \sum_{k=0}^{\infty} \frac{T^k t^k}{k!},$$

called the operator exponential, converges for every $T \in \mathcal{L}(V)$ and for every $t \in \mathbb{R}$.

Proof. Let $T \in \mathcal{L}(V,V)$ and $t \in \mathbb{R}$. By [Apostol, 1974, Definition 8.7], we must show that the sequence of partial sums converges; but since V is a

Banach space, it is sufficient to show that the sequence of partial sums is Cauchy. From [Rudin, 1964, p. 164] it is known that the real series

$$\sum_{k=0}^{\infty} \frac{(||T||\,|t|)^k}{k!},$$

converges for all $t \in \mathbb{R}$ which is equivalent to saying that the sequence of partial sums converges. Since every convergent sequence in a normed space is Cauchy, cf. proposition 3.2, for every $\varepsilon' > 0$, we can pick an $N'(\varepsilon') \in \mathbb{N}$ such that for every $n,m \geq N'(\varepsilon')$ with $n > m$, we have

$$\sum_{k=0}^{n} \frac{(||T||\,|t|)^k}{k!} - \sum_{k=0}^{m} \frac{(||T||\,|t|)^k}{k!} = \sum_{k=m+1}^{n} \frac{(||T||\,|t|)^k}{k!} < \varepsilon'.$$

Now pick an arbitrary $\varepsilon > 0$ and suppose $\varepsilon' := \varepsilon$. By using the above let $N(\varepsilon) = N'(\varepsilon')$. For $n,m \geq N'(\varepsilon')$ assume that $n > m$

$$\begin{aligned} \left|\left| \sum_{k=0}^{n} \frac{T^k t^k}{k!} - \sum_{k=0}^{m} \frac{T^k t^k}{k!} \right|\right| &= \left|\left| \sum_{k=m+1}^{n} \frac{T^k t^k}{k!} \right|\right| \\ &\leq \sum_{k=m+1}^{n} \left|\left| \frac{T^k t^k}{k!} \right|\right|, \end{aligned}$$

by the triangle inequality. Applying proposition 5.2 we see that

$$\sum_{k=m+1}^{n} \left|\left| \frac{T^k t^k}{k!} \right|\right| \leq \sum_{k=m+1}^{n} \frac{||T^k||\,|t|^k}{k!} \leq \sum_{k=m+1}^{n} \frac{||T||^k\,|t|^k}{k!} < \varepsilon'.$$

By the choice of $N(\varepsilon)$ above this becomes

$$\left|\left| \sum_{k=0}^{n} \frac{T^k t^k}{k!} - \sum_{k=0}^{m} \frac{T^k t^k}{k!} \right|\right| < \varepsilon,$$

so the sequence of partial sums is Cauchy. Since T and t were arbitrarily chosen the series converges for every $T \in \mathcal{L}(V)$ and every $t \in \mathbb{R}$. $\square$

5.2 The Matrix Exponential

In order to define the operator exponential on operators (matrices) on finite dimensional spaces we state the following useful observation which states that every linear map from a finite-dimensional space V to V is automatically continuous. Therefore $\mathcal{L}(V)$ contains *all* linear maps from V to V.

Proposition 5.4. *Let V be a finite dimensional vector space. A linear map $T : V \to V$ is uniformly continuous.*

Proof. We should show, according to [Apostol, 1974, Definition 4.46]

$$\forall\, \varepsilon > 0,\ \exists\, \delta(\epsilon) > 0\ v,w \in V : ||v - w|| < \delta \Rightarrow ||T(v) - T(w)|| < \varepsilon.$$

Choose any basis $(v_1, \ldots, v_N)$ for V (this is possible since V by assumption is finite dimensional). A vector $v \in V$ can now be represented as

$$v = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_N v_N, \quad \mu_1, \ldots, \mu_N \in \mathbb{R}.$$

We will use the ℓ_1-norm to establish continuity (by equivalence of norms finite dimensional space, cf. theorem 3.5, any norm will do).

Choose $\varepsilon > 0$, let $M := \max_{1 \le i \le N} ||T(v_i)||$ and set $\delta := \varepsilon / M$. Now, pick $v,w \in V$ such that

$$||v - w|| = |\mu_1 - \lambda_1| + \cdots + |\mu_N - \lambda_N| < \delta.$$

We then have

$$\begin{aligned} ||T(v) - T(w)|| &= ||T(v - w)|| \\ &= \left\| \sum_{i=1}^{N} (\mu_i - \lambda_i) T(v_i) \right\| \\ &\le \sum_{i=1}^{N} |\mu_i - \lambda_i|\, ||T(v_i)|| \\ &\le M \sum_{i=1}^{N} |\mu_i - \lambda_i| < M\delta = \varepsilon. \end{aligned}$$

Since our choice of basis was arbitrary, T is uniformly continuous. □

Given a basis for $\mathbb{R}^N$, the space of all real $N \times N$ matrices and $\mathcal{L}(\mathbb{R}^N)$ are isomorphic (cf. [Axler, 1997, Proposition 3.19]). For the rest of this chapter, we may assume w.l.o.g. that we are working in the natural basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N)$.

Definition 5.3 (Matrix exponential). *For any real $N \times N$ matrix $\boldsymbol{A}$, the matrix exponential* $\exp(\boldsymbol{A}t)$ *is defined as*

$$\exp(\boldsymbol{A}t) := \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!}, \quad t \in \mathbb{R}. \tag{5.1}$$

with

$$\exp(\boldsymbol{A}0) := \boldsymbol{I},$$

where $\boldsymbol{I}$ *is the identity matrix.*

By proposition 5.3, the matrix exponential converges for any $N \times N$ matrix and any choice of t.

Proposition 5.5. *The matrix exponential* $\exp(\boldsymbol{A}t)$*, where* $\boldsymbol{A}$ *is a real* $N \times N$ *matrix and* $t \in \mathbb{R}$*, has the following properties*

(i) $\exp(\boldsymbol{A}t)\exp(\boldsymbol{A}s) = \exp(\boldsymbol{A}(t+s))$.

(ii) $\frac{d}{dt}\exp(\boldsymbol{A}t) = \boldsymbol{A}\exp(\boldsymbol{A}t)$.

Proof. Property (i): We will only sketch the proof. Using the binomial theorem, [Rosen, 1999, p. 256], we see that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k (t+s)^k}{k!} &= \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{\boldsymbol{A}^k}{k!}\frac{k!}{i!(k-i)!} t^i s^{k-i} \\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{\boldsymbol{A}^i \boldsymbol{A}^{k-i}}{i!(k-i)!} t^i s^{k-i} \\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{\boldsymbol{A}^i}{i!} t^i \frac{\boldsymbol{A}^{k-i}}{(k-i)!} s^{k-i}.
\end{aligned}$$

In $\mathbb{R}$ the above is called the Cauchy product of two series, and in [Betounes, 2001, Theorem C.4] it is noted that this can be extended to series of matrices. Since

$$\sum_{k=0}^{\infty} \left|\left|\frac{\boldsymbol{A}^k t^k}{k!}\right|\right| \le \sum_{k=0}^{\infty} \frac{||\boldsymbol{A}||^k |t|^k}{k!},$$

the series on the left-hand above is bounded above and increasing, the above converges (when the left-hand side of the above converges for a series, the series is called absolutely convergent), we can by [Betounes, 2001, Theorem C.4] conclude that the Cauchy product is equal to the product of the respective limits, i.e.

$$\exp(\boldsymbol{A}t)\exp(\boldsymbol{A}s) = \exp(\boldsymbol{A}(t+s)).$$

Property (ii): Using the triangle inequality and the properties of a norm we see that

$$\left|\left|\sum_{k=0}^{n} \frac{\boldsymbol{A}^k t^k}{k!}\right|\right| \le \sum_{k=0}^{n} \frac{||\boldsymbol{A}||^k |t|^k}{k!}$$

for every $n \geq 1$. Since we established that both partial sums converge it follows from [Apostol, 1974, Theorem 8.3] and [Apostol, 1974, Theorem 8.4] that

$$\left|\left|\sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!}\right|\right| \leq \sum_{k=0}^{\infty} \frac{||\boldsymbol{A}||^k |t|^k}{k!}.$$

Now subtracting the two first elements of the series we see that

$$\begin{aligned} ||\exp(\boldsymbol{A}t) - \boldsymbol{I} - \boldsymbol{A}t|| &= \left|\left|\sum_{k=2}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!}\right|\right| \\ &\leq \sum_{k=2}^{\infty} \frac{||\boldsymbol{A}||^k |t|^k}{k!} \\ &= \mathrm{e}^{||\boldsymbol{A}||\,|t|} - 1 - ||\boldsymbol{A}||\,|t|. \end{aligned}$$

Remembering that $\exp(\boldsymbol{A}0) = \boldsymbol{I}$, $\mathrm{e}^{||\boldsymbol{A}||0} = 1$, and dividing by $|t|$ this becomes the definition of the derivative at $t = 0$, i.e.

$$\left|\left|\frac{\exp(\boldsymbol{A}t) - \exp(\boldsymbol{A}0)}{t} - \boldsymbol{A}\right|\right| \leq \frac{\mathrm{e}^{||\boldsymbol{A}||\,|t|} - \mathrm{e}^{||\boldsymbol{A}||0}}{|t|} - ||\boldsymbol{A}||,$$

if we let $t \to 0$. Now remembering that $\frac{d}{dt}\left(\mathrm{e}^{||\boldsymbol{A}||\,|t|}\right) = ||\boldsymbol{A}||$ at $t = 0$, we have that the right-hand side converges to zero, but this also means that the left-hand side also converges to zero, and therefore that

$$\frac{d}{dt}\left(\exp(\boldsymbol{A}t)\right) = \boldsymbol{A},$$

at $t = 0$.

For $t \neq 0$ we use property (i) to see that

$$\begin{aligned} &\left|\left|\frac{\exp(\boldsymbol{A}(t+\varepsilon)) - \exp(\boldsymbol{A}t)}{\varepsilon} - \boldsymbol{A}\exp(\boldsymbol{A}t)\right|\right| = \\ &\qquad = \left|\left|\frac{\exp(\boldsymbol{A}\varepsilon) - \exp(\boldsymbol{A}0)}{\varepsilon} - \boldsymbol{A}\right|\right| \, ||\exp(\boldsymbol{A}t)|| \end{aligned}$$

converges to zero as $\varepsilon \to 0$. This means that in general

$$\frac{d}{dt}\left(\exp(\boldsymbol{A}t)\right) = \boldsymbol{A}\exp(\boldsymbol{A}t).$$

$\square$

In general, given some $N \times N$ matrix $\boldsymbol{A}$, it is tedious if not impossible to apply the definition of $\exp(\boldsymbol{A}t)$ to compute the matrix exponential. There exists general methods that apply to diagonalisable matrices and give an explicit formula of $\exp(\boldsymbol{A}t)$ — but since we are only dealing with very simple matrices, the brute force method is clearly the most satisfactory.

5.3 Determining Explicit Solutions

Now if $T \in \mathcal{L}(\mathbb{R}^N)$, then by proposition 5.4 T is continuous, and by [Berg, 1997, Sætning 4.6] T satisfies a global Lipschitz condition, so T satisfies the conditions of theorem 4.1. We now know that there exists a unique solution around the initial time.

The following proposition assumes that the initial condition is of the form $(0,\boldsymbol{x}_0)$. We will prove in corollary 6.1 that this is sufficient for autonomous systems.

Proposition 5.6 (Explicit solution). *Let $\boldsymbol{A}$ be a real $N \times N$ matrix and let $\boldsymbol{b} \in \mathbb{R}^N$. Define the IVP*

$$\left.\begin{aligned} \boldsymbol{x}' &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0. \end{aligned}\right\} \tag{5.2}$$

Then there exists a unique solution $\boldsymbol{\alpha}$ given by

$$\boldsymbol{\alpha}(t) = \mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{x}_0 + \int_0^t \mathrm{e}^{\boldsymbol{A}s}\,\boldsymbol{b}\,ds,$$

defined for all t in some interval of times containing 0.

Proof. Existence and uniqueness follows from theorem 4.1, since by the remark earlier on this page, $\boldsymbol{A}$ satisfies a global Lipschitz condition and by proposition 5.4 determines a continuous mapping. By differentiation of $\boldsymbol{\alpha}$ we get

$$\begin{aligned} \boldsymbol{\alpha}'(t) &= \boldsymbol{A}\,\mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{x}_0 + \mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{b} \\ &= \boldsymbol{A}(\boldsymbol{\alpha}(t) - \int_0^t \mathrm{e}^{\boldsymbol{A}s}\,\boldsymbol{b}\,ds) + \mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{b} \\ &= \boldsymbol{A}\boldsymbol{\alpha}(t) - \int_0^t \boldsymbol{A}\,\mathrm{e}^{\boldsymbol{A}s}\,\boldsymbol{b}\,ds + \mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{b}, \end{aligned}$$

and by using the first property of the matrix exponential we see that

$$\begin{aligned}\boldsymbol{\alpha}'(t) &= \boldsymbol{A}\boldsymbol{\alpha}(t) - (\mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{b} - \boldsymbol{I}\boldsymbol{b}) + \mathrm{e}^{\boldsymbol{A}t}\,\boldsymbol{b}\\ &= \boldsymbol{A}\boldsymbol{\alpha}(t) + \boldsymbol{b}.\end{aligned}$$

Hence $\boldsymbol{\alpha}$ is a solution to the first-order system. For $t = 0$ we have

$$\boldsymbol{\alpha}(0) = \mathrm{e}^{\boldsymbol{A}0}\,\boldsymbol{x}_0 + \int_0^0 \mathrm{e}^{\boldsymbol{A}s}\,\boldsymbol{b}\,ds = \boldsymbol{I}\boldsymbol{x}_0 = \boldsymbol{x}_0,$$

so by theorem 4.1, $\boldsymbol{\alpha}$ is the unique solution to the IVP in some interval of times containing 0. □

It can be proved that solutions to IVPs of the form above are defined for all $t \in \mathbb{R}$ [Betounes, 2001, p. 162]. When applied to our own system, this result is nevertheless not necessary, since we have already proved in section 4.3.1 that any solution to our system is defined for all $t \geq 0$.

5.3.1 Explicit Solutions in the Linear Region

We will now consider the problem of determining explicit solutions to our system (2.3) on p. 11). For the case $\boldsymbol{z} \in ([2;\infty[\times\mathbb{R}) \cup (]-\infty;-2[\times\mathbb{R})$ the force function $F(x)$ is zero, so we can directly determine solutions from the original equation (1.3), since we have

$$\frac{d^2x}{dt^2}(t) = 0, \quad t \geq 0,$$

which is clearly solved by $x(t) = at + b$, $a,b \in \mathbb{R}$ with $a = v(0)$ and $b = x(0)$. This readily gives the solution for the corresponding IVP (first-order system), namely

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \boldsymbol{x}_0. \tag{5.3}$$

This trivial case is merely motion with constant velocity, in correspondence with the fact that there is no force present (Newton's first law).

Remark. We have made no restrictions on the velocity in the above — but it is clear that if the initial velocity is directed towards the origin, the solution will only be valid in a limited interval of times. The same observation applies to the following discussion.

Somewhat less trivial are the cases where $\boldsymbol{z} \in (]-2;1] \times \mathbb{R}) \cup (]1;2] \times \mathbb{R})$. We shall discuss the case $\boldsymbol{z} \in (]1;2] \times \mathbb{R})$, the other is similar.

In this particular region, we have the linear system

$$\boldsymbol{z}' = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}, \tag{5.4}$$

where the $\boldsymbol{A}$ and $\boldsymbol{b}$ are given by

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \tag{5.5}$$

Proposition 5.7. *Let* $\boldsymbol{A}$ *be given as in* (5.5). *Then for* $n \geq 0$

$$\boldsymbol{A}^n = \begin{bmatrix} \frac{1}{2}(1+(-1)^n) & \frac{1}{2}(1-(-1)^n) \\ \\ \frac{1}{2}(1-(-1)^n) & \frac{1}{2}(1+(-1)^n) \end{bmatrix} := \begin{bmatrix} b_n & a_n \\ a_n & b_n \end{bmatrix}.$$

Proof. If $n = 0$, the formula clearly holds, since $\boldsymbol{A}^0 = I$; the remaining cases follows by induction. For $n = 1$, we see that

$$a_1 = 0, \text{ and } b_n = 1,$$

so the formula is valid.

Now, assume the formula holds for $n > 1$. We then have

$$\boldsymbol{A}\boldsymbol{A}^n = \begin{bmatrix} b_1 & a_1 \\ a_1 & b_1 \end{bmatrix} \begin{bmatrix} b_n & a_n \\ a_n & b_n \end{bmatrix} = \begin{bmatrix} b_1 b_n + a_1 a_n & b_1 a_n + a_1 b_n \\ a_1 b_n + b_1 a_n & a_1 a_n + b_1 b_n \end{bmatrix}.$$

But

$$b_1 b_n + a_1 a_n = a_1 a_n = a_n = b_{n+1}, \text{ and } b_1 a_n + a_1 b_n = a_1 b_n = b_n = a_{n+1}$$

so

$$\boldsymbol{A}^{n+1} = \begin{bmatrix} b_{n+1} & a_{n+1} \\ a_{n+1} & b_{n+1} \end{bmatrix},$$

which completes the proof. □

Proposition 5.8. *Let* $\boldsymbol{A}$ *be given as in* (5.5). *Then*

$$\exp(\boldsymbol{A}t) = \begin{bmatrix} \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t}) & \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t}) \\ \\ \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t}) & \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t}) \end{bmatrix}.$$

Proof. This is done simply by calculations. We have

$$\begin{aligned}
\exp(\boldsymbol{A}t) &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{2}(1+(-1)^k)\frac{t^k}{k!} & \sum_{k=0}^{\infty} \frac{1}{2}(1-(-1)^k)\frac{t^k}{k!} \\ \\ \sum_{k=0}^{\infty} \frac{1}{2}(1-(-1)^k)\frac{t^k}{k!} & \sum_{k=0}^{\infty} \frac{1}{2}(1+(-1)^k)\frac{t^k}{k!} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \sum_{k=0}^{\infty}(\frac{t^k}{k!} + \frac{(-t)^k}{k!}) & \sum_{k=0}^{\infty}(\frac{t^k}{k!} - \frac{(-t)^k}{k!}) \\ \\ \sum_{k=0}^{\infty}(\frac{t^k}{k!} - \frac{(-t)^k}{k!}) & \sum_{k=0}^{\infty}(\frac{t^k}{k!} + \frac{(-t)^k}{k!}) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t}) & \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t}) \\ \\ \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t}) & \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t}) \end{bmatrix}.
\end{aligned}$$

In the above we recognise the infinite sums as being a certain cases of the power series representation of the exponential function, cf. [Rudin, 1964, p. 163]. □

According to the formula given in proposition 5.6, we can now obtain the solution to linear system. Let us first calculate the following

$$\begin{aligned}
\exp(\boldsymbol{A}t)\boldsymbol{b} &= \begin{bmatrix} \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t}) & \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t}) \\ \\ \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t}) & \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t}) \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{e}^t + \mathrm{e}^{-t} \\ -\mathrm{e}^t - \mathrm{e}^{-t} \end{bmatrix}.
\end{aligned}$$

Hence

$$\int_0^t \exp(\boldsymbol{A}s)\boldsymbol{b}\,ds = \begin{bmatrix} -\mathrm{e}^t - \mathrm{e}^{-t} + 2 \\ -\mathrm{e}^t + \mathrm{e}^{-t} \end{bmatrix}.$$

With this in hand, we obtain the explicit solution

$$\begin{aligned}
\boldsymbol{\alpha}(t) &= \boldsymbol{x}_0 \exp(\boldsymbol{A}t) + \int_0^t \exp(\boldsymbol{A}s)\boldsymbol{b}\,ds \\
&= \frac{1}{2}\begin{bmatrix} \mathrm{e}^t + \mathrm{e}^{-t} & \mathrm{e}^t - \mathrm{e}^{-t} \\ \\ \mathrm{e}^t - \mathrm{e}^{-t} & \mathrm{e}^t + \mathrm{e}^{-t} \end{bmatrix} \boldsymbol{x}_0 + \begin{bmatrix} -\mathrm{e}^t - \mathrm{e}^{-t} + 2 \\ -\mathrm{e}^t + \mathrm{e}^{-t} \end{bmatrix}.
\end{aligned}$$

Together with the result on the region $x(t) > 2$, it is now possible to completely describe the trajectories for instance in cases with positive energy.

This is merely a matter of piecing the two curves for the two different regions together with the correct initial conditions and will not be done here.

From a qualitative point of view, the explicit solutions say only a little about the behaviour of the system. However, the simple result that positive velocity in the regions $x \leq -2$, $x > 2$ yields motion with constant velocity is actually an important observation which will prove useful in the study of stability which follows in section 6.2.

CHAPTER 6

Dynamics of our System

In this chapter we will focus specifically on the dynamics of our own system, respectively as a second-order system, (1.3) on p. 1, and as a system of first-order DEs, (2.3) on p. 11 — depending on which representation gives the most intuitive framework.

The analysis will be based on the general properties of autonomous systems of differential equations. Recall that an autonomous system of differential equation is a system involving a vector field $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ that does not depend on time

$$\boldsymbol{x}' = \boldsymbol{X}(\boldsymbol{x}). \tag{6.1}$$

Assuming that $\boldsymbol{X}$ is continuous and satisifes a local Lipschitz condition, all the previous results on existence and uniqueness apply to this particular case.

6.1 Properties of Autonomous Systems

The following section, based on [Betounes, 2001, Section 3.5, 7.1], presents and proves a few simple but important general properties of autonomous systems. Furthermore we will introduce the *flow* of an autonomous vector field, a concept which is useful in the following investigation of the stability problem.

Notation. Suppose I is an open interval and $c \in I$. Then we define

$$I - c :=]\inf I - c; \sup I - c[.$$

Also we will use the term *an integral curve of a vector field* meaning a curve which solves an IVP generated by a vector field $\boldsymbol{X}$.

Theorem 6.1 (Time translational symmetry). *Let* $\boldsymbol{\alpha} : I \to \mathbb{R}^N$ *be an integral curve of an autonomous vector field* $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$. *Now for all* $t_0 \in I$ *the curve* $\boldsymbol{\beta} : I - t_0 \to \mathbb{R}^N$ *defined by*

$$\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(t + t_0), \quad \forall\, t \in I - t_0,$$

is also an integral curve of $\boldsymbol{X}$.

Proof. Assume that $\boldsymbol{\alpha}(t)$ is an integral curve of $\boldsymbol{X}$. Define $\boldsymbol{\beta}(t) := \boldsymbol{\alpha}(t + t_0)$, for all $t \in I - t_0$. Because of our choice of interval, $\boldsymbol{\beta}$ is well-defined and since $\boldsymbol{\alpha} \in \mathcal{O}$ for all $t \in I$, $\boldsymbol{\beta} \in \mathcal{O}$ for all $t \in I - t_0$. Similarly, $\boldsymbol{\beta}$ must be differentiable for all $t \in I - t_0$ and finally

$$\boldsymbol{\beta}'(t) = \boldsymbol{\alpha}'(t + t_0) = \boldsymbol{X}(\boldsymbol{\alpha}(t + t_0)) = \boldsymbol{X}(\boldsymbol{\beta}(t)), \quad \forall\, t \in I - t_0.$$

Therefore $\boldsymbol{\beta}$ is an integral curve of $\boldsymbol{X}$. □

Corollary 6.1 (Translation property). *Suppose* $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ *is an autonomous vector field. Then the maximal interval of existence* $I_{(t_0, \boldsymbol{x_0})}$, *and the maximal solution* $\boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)} : I_{(t_0,\boldsymbol{x}_0)} \to \mathbb{R}^N$ *for* $\boldsymbol{x}_0 \in \mathcal{O}$ *and* $t \in \mathbb{R}$ *satisfy the the following relations*

$$\begin{aligned} I_{(t_0,\boldsymbol{x}_0)} &= I_{(0,\boldsymbol{x}_0)} + t_0 \\ \boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t) &= \boldsymbol{\alpha}_{(0,\boldsymbol{x}_0)}(t - t_0), \quad \forall\, t \in I_{(t_0,\boldsymbol{x}_0)}. \end{aligned}$$

Proof. Define

$$\boldsymbol{\beta}(t) := \boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t + t_0), \forall\, t \in I_{(t_0,\boldsymbol{x}_0)} - t_0.$$

By theorem 6.1, since $\boldsymbol{X}$ is autonomous, $\boldsymbol{\beta}$ is an integral curve of $\boldsymbol{X}$. Since

$$\boldsymbol{\beta}(0) = \boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(0 + t_0) = \boldsymbol{x}_0,$$

they satisfy the same IVP. From theorem 4.2, we see that

$$\begin{aligned} I_{(t_0,\boldsymbol{x}_0)} - t_0 \subseteq I_{(0,\boldsymbol{x}_0)} &\Rightarrow \\ I_{(t_0,\boldsymbol{x}_0)} \subseteq I_{(0,\boldsymbol{x}_0)} &+ t_0. \end{aligned} \tag{6.2}$$

Hence $\boldsymbol{\beta} = \boldsymbol{\alpha}_{(0,\boldsymbol{x}_0)}$ on $I_{(t_0,\boldsymbol{x}_0)} - t_0$.

From the same idea we have a curve $\boldsymbol{\gamma}$ defined by

$$\boldsymbol{\gamma}(t) := \boldsymbol{\alpha}_{(0,\boldsymbol{x}_0)}(t - t_0), \forall\, t \in I_{(0,\boldsymbol{x}_0)} + t_0,$$

which is an integral curve of $\boldsymbol{X}$ satisfying the same IVP as $\boldsymbol{\alpha}_{(t_0,\boldsymbol{x_0})}$. Again from theorem 4.2

$$I_{(0,\boldsymbol{x}_0)} + t_0 \subseteq I_{(t_0,\boldsymbol{x_0})}, \tag{6.3}$$

and $\boldsymbol{\gamma} = \boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}$ on $I_{(0,\boldsymbol{x}_0)} + t_0$.

By joining (6.2) and (6.3) we get

$$\begin{aligned} &I_{(t_0,\boldsymbol{x}_0)} \subseteq I_{(0,\boldsymbol{x}_0)} + t_0 \subseteq I_{(t_0,\boldsymbol{x}_0)} \Rightarrow \\ &I_{(t_0,\boldsymbol{x}_0)} = I_{(0,\boldsymbol{x}_0)} + t_0, \end{aligned}$$

and thus $\boldsymbol{\alpha}_{(0,\boldsymbol{x}_0)}(t - t_0) = \boldsymbol{\alpha}_{(t_0,\boldsymbol{x}_0)}(t)$ on $I_{(t_0,\boldsymbol{x}_0)}$, which is the result of the corollary. □

In our study of stability it will be convenient to treat solutions of an autonomous differential equation in wider generality, not depending on a specific initial value or time. Corollary 6.1 tells us that for autonomous systems, we can translate the initial condition $(t_0,\boldsymbol{x}_0)$ for some solution into the origin by appropriate translation of the maximal interval of existence.

The definition of *the flow map* for an autonomous system is the the natural way of bundling all integral curves with initial condition $(0,\boldsymbol{x}_0)$ for some initial point $\boldsymbol{x}_0$ together; and treat them as one single object.

Definition 6.1 (Flow map). *Let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ be an autonomous vector field, and let*

$$D := \{(t,\boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^N \mid \boldsymbol{x} \in \mathcal{O} \text{ and } t \in I_{(0,\boldsymbol{x})}\}.$$

Then the flow generated by $\boldsymbol{X}$ is the map $\phi : D \to \mathbb{R}^N$ defined by

$$\phi_t(\boldsymbol{x}) = \boldsymbol{\alpha}_{(0,\boldsymbol{x})}(t). \tag{6.4}$$

Figure 6.1 gives a depiction of, how one could describe the flow map geometrically.

Remark. One can define a similar flow map for non-autonomous with the difference that we must introduce dependence on initial time. However, in connection with stability analysis this is of no interest.

6.1.1 Equilibrium Points and Stability

In chapter 1 we emphasized the significance of the origin and the points $(0, \pm 2)$; these points are so-called *equilibrium points* of the system (1.3), i.e. points which are mapped into the zero vector by the vector field.

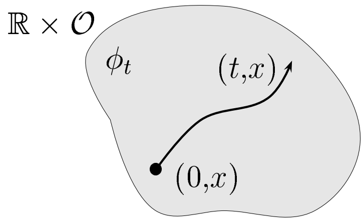

Figure 6.1: *The flow $\phi_t(\boldsymbol{x})$ for an autonomous vector field describes how the point $\boldsymbol{x}$ moves when influenced by the vector field.*

Definition 6.2 (Equilibrium point). *Let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ be an autonomous vector field. A point $\boldsymbol{c} \in \mathcal{O}$ is called an equilibrium point (or a fixed point) of $\boldsymbol{X}$ if*

$$\boldsymbol{X}(\boldsymbol{c}) = \boldsymbol{0}.$$

In our specific case, this of course means (as already noticed in section 1.2) that the force function is zero.

Viewed as the physical particle-force problem, the equilibrium points are special in the sense that if one places the particle in these points with zero velocity, it will remain there for all time.

This is easy to verify in a more precise manner. For instance, the curve $\boldsymbol{\alpha}(t) := (0{,}2)$ clearly solves the system of DEs (2.3) on p. 11 since

$$\boldsymbol{\alpha}'(t) = (0{,}0) = \boldsymbol{X}(\boldsymbol{\alpha}).$$

Also, $\boldsymbol{\alpha}(0) = (0{,}2)$ so by proposition 4.4, $\boldsymbol{\alpha}$ is the unique solution.

The problem of investigating stability of equilibrium points is to determine, what happens if the initial conditions are perturbed slightly from the equilibrium point — will the flow of the vector field generating the system of DEs eventually carry the point away from equilibrium point or will it remain near it for all time? If the last case is true, the equilibrium point is said to be *stable*.

Definition 6.3 (Stability). *Let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ be an autonomous vector field, and let ϕ_t be the flow of $\boldsymbol{X}(\boldsymbol{x})$. An equilibrium point $\boldsymbol{c} \in \mathcal{O}$ is called stable if*

$$\forall\, \varepsilon > 0, \exists\, \delta > 0, \forall\, \boldsymbol{y} \in B(\boldsymbol{c}, \delta), \forall\, t \in I_{(0,\boldsymbol{y})} \cap \mathbb{R}_0^+ : ||\phi_t(\boldsymbol{y}) - \boldsymbol{c}|| < \varepsilon.$$

An equilibrium point is called unstable if it is not stable.

According to definition 6.3, we can obtain a criterion for instability by simply denying the criterion for stability. We obtain, that $\boldsymbol{c}$ is unstable if

$$\exists\, \varepsilon_0 > 0 : \forall\, \delta > 0,\, \exists\, \boldsymbol{y} \in B(\boldsymbol{c}, \delta), \exists\, t_0 \in I_{(0,\boldsymbol{y})} \cap \mathbb{R}_0^+ : ||\phi_{t_0}(\boldsymbol{y}) - \boldsymbol{c}|| \geq \varepsilon_0. \quad (6.5)$$

This definition is equivalent to a more practical definition stated in terms of sequences converging to the equilibrium point.

Proposition 6.1 (Condition for instability). *Let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^N$ be an autonomous vector field, and let ϕ_t be the flow of $\boldsymbol{X}$. An equilibrium point $\boldsymbol{c} \in \mathcal{O}$ is unstable if, and only if, there exists $\varepsilon_0 > 0$, a sequence $\{\boldsymbol{x}_n\}_{n\geq 1}$ of points, and a sequence $\{t_n\}_{n\geq 1}$ of times such that $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{c}$ and*

$$||\phi_{t_n}(\boldsymbol{x}_n) - \boldsymbol{c}|| \geq \varepsilon_0, \quad n \geq 1.$$

Proof. An equilibrium point $\boldsymbol{c}$ is unstable if it is not stable, that is, if (6.5) is satisfied. Assume first that $\boldsymbol{c}$ is unstable. Then there exists $\varepsilon_0 > 0$ satisfying the definition of instability. Now, define $\{\boldsymbol{x}_n\}_{n\geq 1}$ by

$$\boldsymbol{x}_n := \boldsymbol{y}_n \in B\left(\boldsymbol{c}, \frac{\varepsilon_0}{n}\right) \text{ such that } \exists\, t \in I_{(0,\boldsymbol{y}_n)} \cap \mathbb{R}_0^+ \text{ so } ||\phi_t(\boldsymbol{y}_n) - \boldsymbol{c}|| \geq \varepsilon_0.$$

By assumption, there exists such points $\boldsymbol{y}_n$ for every ball $B(\boldsymbol{c},\delta)$, $\delta > 0$, so the sequence is well-defined but not necessarily unique. Similarly, define

$$t_n := t_0 \in I_{(0,\boldsymbol{y}_n)} \cap \mathbb{R}_0^+ : ||\phi_{t_0}(\boldsymbol{x}_n) - \boldsymbol{c}|| \geq \varepsilon_0.$$

By assumption, t_n also exists. We have now constructed a sequence $\{\boldsymbol{x}_n\}_{n\geq 1}$ of points and a sequence $\{t_n\}_{n\geq 1}$ of times such that the flow always leaves the open ball $B(\boldsymbol{c},\varepsilon)$. We need only verify that $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{c}$. However, since $||\boldsymbol{x}_n - \boldsymbol{c}|| < \varepsilon_0/n$ and $\lim_{n\to\infty} \varepsilon_0/n = 0$, we have

$$\lim_{n\to\infty} ||\boldsymbol{x}_n - \boldsymbol{c}|| = 0 \Leftrightarrow \lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{c}.$$

Conversely, suppose there exists an $\varepsilon_0' > 0$, a sequence of points $\{\boldsymbol{x_n}\}_{k\geq 1}$ and a sequence of times $\{t_k\}_{k\geq 1}$ such that $\lim_{k\to\infty} \boldsymbol{x}_k = \boldsymbol{c}$ and

$$||\phi_{t_k}(\boldsymbol{x}_k) - \boldsymbol{c}|| \geq \varepsilon_0', \quad \forall\, k \geq 1.$$

We must prove that $\boldsymbol{c}$ is unstable. Choose $\delta' > 0$. By convergence of $\boldsymbol{x}_n$ we have

$$\forall\, \delta' > 0, \exists\, N(\delta') \in \mathbb{N} : m \geq N(\delta') \Rightarrow ||\boldsymbol{x}_m - \boldsymbol{c}|| < \delta'.$$

Thus for any ball of radius δ' centered around $\boldsymbol{c}$ we can find at least one point $\boldsymbol{x}_m$ in the sequence, such that the definition of instability is satisfied. This is sufficient and we conclude that $\boldsymbol{c}$ is unstable. $\square$

6.2 Stability of Equilibrium Points

The following theorem on stability of the origin in our system, (2.3) on p. 11, simply applies the definition of stability. While this may not be the most elegant approach, it is simple and efficient in our case due to the important fact that there exists an energy function.

Theorem 6.2 (Stability of origin). *The origin is a stable equilibrium point of the system* (2.3) *on p. 11.*

Proof. Throughout this proof, we will use the usual Euclidean norm. Fix $\varepsilon > 0$, and pick $\delta := \min\left\{\frac{1}{\sqrt{2}}, \frac{\varepsilon^2}{2\sqrt{2}}\right\}$. The strategy will be to use the energy function to show that both position and velocity can be bounded sufficiently enough to ensure that the trajectories will remain in any ball of radius ε. Our strategy is now to prove that with this choice of δ, the energy is less than a certain constant. Now choose $(a, b) \in B(\mathbf{0},\delta)$. We have

$$\begin{aligned} ||(a,b)|| < \delta &\Leftrightarrow \\ \sqrt{a^2+b^2} < \delta &\Rightarrow \\ \frac{a^2}{2} + \frac{b^2}{2} < \frac{\delta^2}{2} &\Rightarrow \\ \frac{b^2}{2} + \frac{a^4}{4} - \frac{3}{4} < \frac{\delta^2}{2} - \frac{3}{4}&, \end{aligned}$$

where we use that since $|a| < \delta < 1$ so that $a^4/4 < a^2/2$. Since $V(a) = a^4/4 - 3/4$ when $|a| < 1$, the left-hand side in the last inequality above is in fact the energy function evaluated in (a,b). Now because of the conservation of energy, cf. section 1.1.2, the inequality holds for all $t \geq 0$. In other words, if we let $\phi_t((a,b)) := (q(t),\dot{q}(t))$, we have limited the energy as

$$\tilde{E}(q(t),\dot{q}(t)) = \frac{\dot{q}(t)^2}{2} + V(q(t)) < \frac{\delta^2}{2} - \frac{3}{4} \leq -\frac{1}{2}, \quad t \geq 0. \tag{6.6}$$

In order to simplify the inequality, we would like to find an expression for $V(q(t))$. This is possible if the position is sufficiently restricted, say $|q(t)| < 1$. This is in fact the case — to show this we assume that $|q(t)| \geq 1$ whenever $t \geq \tau$ for some $\tau \geq 0$, and obtain a contradiction. By symmetry we have $V(|q(t)|) = V(q(t))$, so when $|q(t)| \geq 1$

$$V(1) = -\frac{1}{2} \leq V(q(t)) \leq 0.$$

But using (6.6) we see that

$$V(q(t)) < -\frac{1}{2} - \frac{\dot{q}(t)^2}{2} \Rightarrow$$
$$-\frac{\dot{q}(t)^2}{2} > V(q(t)) + \frac{1}{2} \geq 0.$$

This implies that $\dot{q}(t)^2 < 0$, a contradiction, and we conclude that $|q(t)| < 1$ for all $t \geq 0$.

Using the definition of the potential function in the region $-1 < q(t) < 1$, we can now reduce (6.6) to

$$\frac{\dot{q}(t)^2}{2} + \frac{q(t)^4}{4} - \frac{3}{4} < \frac{\delta^2}{2} - \frac{3}{4}.$$

By conservation of energy, we see that $\dot{q}(t)$ attains its maximum value when $q(t) = 0$, so

$$\frac{\dot{q}(t)^2}{2} - \frac{3}{4} < \frac{\delta^2}{2} - \frac{3}{4} \Leftrightarrow \dot{q}(t)^2 < \delta^2.$$

Since $\delta < 1/\sqrt{2} < 1$ by assumption, the following is also true

$$\frac{\dot{q}(t)^2}{\sqrt{2}} < \delta. \tag{6.7}$$

Similarly $q(t)$ attains its maximum value when $\dot{q}(t) = 0$, so

$$\frac{q(t)^4}{4} - \frac{3}{4} < \frac{\delta^2}{2} - \frac{3}{4} \Rightarrow \frac{q(t)^2}{\sqrt{2}} < \delta. \tag{6.8}$$

Now adding (6.7) and (6.8) we get

$$\frac{\dot{q}(t)^2}{\sqrt{2}} + \frac{q(t)^2}{\sqrt{2}} < 2\delta \leq 2\frac{\varepsilon^2}{2\sqrt{2}} \Rightarrow \sqrt{\dot{q}(t)^2 + q(t)^2} < \varepsilon.$$

Since ε was arbitrarily chosen, $\phi_t((a,b)) \in B(\mathbf{0},\varepsilon)$ for all $t \in \mathbb{R}_0^+$ and all $\varepsilon > 0$. Also, $(a,b) \in B(\mathbf{0},\delta)$ was arbitrary, so according to definition 6.3, the origin is a stable equilibrium point. □

The following proof of instability is based on proposition 6.1 and hence requires us to find the flow of a sequence converging to the fixed point(s). In the regions $x \leq -2$ and $x > 2$ this is, however, easy. Recall from p.61 that the solution (and therefore the flow) e.g. in the region $x > 2$ was given by

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \boldsymbol{z} = \phi_t(\boldsymbol{z}).$$

The definition of instability in terms of sequences, cf. proposition 6.1, will prove useful since the above formula for the flow is not valid when $\boldsymbol{z} = (2,v)$ for any $v \in \mathbb{R}$. However, we may just choose a sequence with flow as given above, converging to this point, and apply proposition 6.1.

Theorem 6.3 (Unstable fixed points). *The points* $(p,0)$ *and* $(-p,0)$ *for* $p \geq 2$ *are unstable equilibrium points of the system* (2.3) *on p. 11.*

Proof. For the following, let $||\cdot||_1$ be the ℓ_1 norm on $\mathbb{R}^2$. For the point $(p,0)$ define the sequence $\{\boldsymbol{x}_n\}_{n\geq 1}$ by $\boldsymbol{x}_n = (p+1/n, 1/n)$, and let the sequence $\{t_n\}_{n\geq 1}$ be given by $t_n = n$.

This gives a sequence of initial values $(t_0,\boldsymbol{x}_0) = (0,\boldsymbol{x}_n)$ and by the observation earlier on this page we have the following formula for the flow of $\boldsymbol{x}_n$

$$\phi_{t_n}(\boldsymbol{x}_n) = \left(p + \frac{1}{n}t_n + \frac{1}{n}, \frac{1}{n}\right),$$

when $\phi_{t_n}(\boldsymbol{x}_n) \in]2,\infty[\times\mathbb{R}^+$, but since $t_n \geq 1$ for all $n \geq 1$ this is ensured.

We must now find an ε such that for any choice of δ, the flow $\phi_{t_n}(\boldsymbol{x}_n)$ when restricted to a ball of radius δ centred at p, leaves the ε-ball centred at $(0,p)$ at some time.

Now choose $\varepsilon_0 = 1$, fix $\delta > 0$, and pick $n_\delta = \left\lceil \frac{2}{\delta} \right\rceil + 1$. We then have

$$\begin{aligned} n_\delta > \frac{2}{\delta} &\Leftrightarrow \\ \frac{1}{n_\delta} + \frac{1}{n_\delta} < \delta &\Leftrightarrow \\ ||\,(p,0) - (p + \tfrac{1}{n_\delta}, \tfrac{1}{n_\delta})||_1 < \delta &\Rightarrow \\ \boldsymbol{x}_{n_\delta} \in B((p,0),\delta)& \end{aligned}$$

and

$$\phi_{t_{n_\delta}}(\boldsymbol{x}_{n_\delta}) = \left(p + \frac{1}{n_\delta}n_\delta + \frac{1}{n_\delta}, \frac{1}{n_\delta}\right) = \left(p + 1 + \frac{1}{n_\delta}, \frac{1}{n_\delta}\right)$$

We now see that

$$\begin{aligned} \left\|\phi_{t_{n_\delta}}(\boldsymbol{x}_{n_\delta}) - (p,0)\right\|_1 &= \left\|\left(p + 1 + \frac{1}{n_\delta}, \frac{1}{n_\delta}\right) - (p,0)\right\|_1 \\ &= \left\|\left(1 + \frac{1}{n_\delta}, \frac{1}{n_\delta}\right)\right\|_1 > 1 = \varepsilon_0. \end{aligned}$$

so $\phi_{t_{n_\delta}}(\boldsymbol{x}_{n_\delta}) \notin B((p,0),\varepsilon_0)$.

By defining the sequence $\{\boldsymbol{x}_n\}_{n\geq 1}$ by $\boldsymbol{x}_n = \left(-p, -\frac{1}{n}\right)$ for the set of points $(-p,0)$, the proof follows similarly. ☐

6.3 Periodicity of Certain Solutions

Recall from chapter 1 that there exists a constant of motion for the equation (1.3), which is the *total mechanical energy* of the system.

Indeed, if q solves (1.3) on p. 1 then

$$\tilde{E}(q,\dot{q}) = \frac{\dot{q}^2}{2} + V(q) = \text{constant}.$$

In chapter 1 we saw that the integral curves for the corresponding first-order system for which we had negative mechanical energy were seemingly closed, corresponding to periodic movement.

Since we now know that a unique solution to the initial value problem generated by (1.3) always exists, it makes sense to ask whether solutions with negative energy really are periodic functions.

Definition 6.4 (Periodic function). *Suppose S is a non-empty set. A function $f : \mathbb{R} \to S$ is said to be periodic if*

$$\exists\, \tilde{T} > 0 : f(t) = f(t + \tilde{T}), \quad \forall\, t \in \mathbb{R}.$$

If f is periodic, then we define the period T of f as

$$T = \inf\{\tilde{T} \,:\, f(t) = f(t + \tilde{T}), \quad \forall\, t \in \mathbb{R}\}.$$

One can, if desired, define periodic functions on any interval which has no finite upper bound — but the definition does not apply if the interval has finite upper bound. We will primarily be concerned with periodic functions defined on $\mathbb{R}_0^+$.

Since we are dealing with an autonomous system, it turns out to be sufficient to show that there exists some $T > 0$ such that the position and velocity at time T is equal to the initial position and velocity.

Theorem 6.4 (Condition for periodicity). *Suppose that $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^N$ is the maximal solution to an autonomous IVP. If there exists some $T > 0$ such that*

$$\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(T),$$

then $\boldsymbol{\alpha}$ is periodic.

Proof. Assume that there exists some positive real number T such that $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(T)$. By time translational symmetry of autonomous systems, theorem 6.1, the function

$$\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(t+T), \quad t \in \mathbb{R},$$

is also a solution and since $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(T) = \boldsymbol{\beta}(0)$, by existence and uniqueness the solutions must be the equal at all points $t \in \mathbb{R}$. Thus, $\boldsymbol{\alpha}$ is periodic. $\square$

Remark. For our case we wish to show that if q solves (1.3), then q is periodic. However, if we can show that $\boldsymbol{\alpha}(t) = (q(t),\dot{q}(t))$ is also periodic, so is q.

Before we prove periodicity of the solutions satisfying $\tilde{E}(q,\dot{q}) < 0$, we need the following useful observation.

Lemma 6.1. *Any solution q to an IVP generated by* (1.3) *for which the energy $\tilde{E}(q,\dot{q}) < 0$ satisfies $-2 < q(t) < 2$ for every $t \geq 0$.*

Proof. Assume the contrary; that there exists some τ such that $q(\tau) \geq 2$. Then $V(q(\tau)) = 0$ and we conclude

$$\tilde{E}(q,\dot{q}) = \frac{\dot{q}^2}{2} + V(q(\tau)) = \frac{\dot{q}^2}{2} < 0$$

which is impossible. $\square$

Theorem 6.5 (Periodicity of solutions). *Let $q : \mathbb{R}_0^+ \to \mathbb{R}$ be the solution to the IVP generated by* (1.3)*, satisfying $q(0) = 0$ and*

$$\tilde{E}(q,\dot{q}) < 0.$$

Then q is periodic.

Proof. Let q denote an arbitrary solution which satisfies the assumptions and assume $\dot{q}(0) = a > 0$. The proof is similar for $a < 0$; the case $a = 0$ is trivial since $q(t) \equiv 0$ (by existence and uniqueness) and therefore periodic. Since $\dot{q}(0) = a > 0$, q must increase on some small interval of times

$$\exists\, \delta > 0 : t \in]0;\delta] \Rightarrow q(t) > 0. \tag{6.9}$$

Let

$$\tau_1 := \sup\{t \geq \delta \mid \dot{q}(t) > 0\}. \tag{6.10}$$

We must prove that $\tau_1 < \infty$ (in other words, that there exists some time where the velocity is zero). Set

$$\xi := \sup_{t \geq \delta} F(q(t)).$$

By (6.9) and lemma 6.1, we see that $0 < q(\delta) \leq q(t) < 2$ when $\delta \leq t < \tau_1$. This implies $\xi < 0$.

We can now make the following estimate, using the order preservation properties of the Riemann integral [Apostol, 1974, theorem 7.20]

$$\dot{q}(t) = a + \int_0^t \ddot{q}(s)ds \leq a + \int_0^t \xi ds = a + \xi t, \qquad \text{whenever } t \geq \delta.$$

Since $a + \xi t \to -\infty$ as $t \to \infty$, and since $\dot{q}$ is continuous we can apply [Apostol, 1974, Theorem 4.32 (Bolzano)], and conclude that τ_1 in (6.10) must be finite and consequently $\dot{q}(\tau_1) = 0$.

Also, $\ddot{q}(\tau_1) = F(q(\tau_1)) < 0$; hence $\dot{q}(t)$ must be strictly decreasing, in some interval of times $\tau_1 < t < \tau_2$ satisfying $q(t) > 0$. In fact

$$\tilde{T} = \sup\{t > \tau_1 \,|\, \ddot{q}(t) < 0\},$$

is precisely the time satisfying $q(\tilde{T}) = 0$. When $t = \tilde{T}$, $V(q) = -3/4$ and by energy conservation $\dot{q}(t) = -a$.

We now have a corresponding situation (with reversed signs) and similar reasoning clearly shows that there exists some time T where the particle passes the origin with velocity $\dot{q}(T) = a$

Therefore, there exists $T > 0$ such that $q(0) = q(T)$ and $\dot{q}(0) = \dot{q}(T)$, and applying theorem 6.4 on $\boldsymbol{\alpha}(t) = (q(t),\dot{q}(t))$, we see that $\boldsymbol{\alpha}$ is periodic, so $q(t)$ must also be periodic. □

The above result can be strengthened in the sense that we can prove that *any* solution corresponding to negative energy must be periodic.

Corollary 6.2. *Suppose q solves an IVP generated by* (1.3) *and*

$$\tilde{E}(q,\dot{q}) < 0.$$

Then q is periodic.

Proof. Suppose $q(0) = x_0$, $\dot{q}(0) = v_0$ and let $\tilde{E}(q,\dot{q}) = E_q < 0$. We can assume $x_0 \neq 0$; this case is covered by theorem 6.5.

Construct an IVP from the same differential equation by requiring $x(0) = 0$, $x'(0) = a > 0$. Suppose β is the solution and that $\tilde{E}(\beta,\beta') = E_q$. We now show that there exists some time $s \geq 0$ such that $\beta(s) = x_0$

Assume $x_0 > 0$. By theorem 6.5, there exists an $x_\beta \geq 0$ such that β oscillates between $\pm x_\beta$ satisfying $V(\pm x_\beta) = E_q$. Furthermore, by energy conservation

$V(x_0) \leq V(x_\beta)$ — and since V is strictly increasing for $0 \leq x < 2$, we see that $x_0 \leq x_\beta$. Since β is continuous, we can apply [Apostol, 1974, Theorem 4.33], the intermediate value theorem for continuous functions, and conclude

$$\exists\, s \geq 0 : \beta(s) = x_0.$$

When $\beta(s) = x_0$, by energy conservation we must have $\beta'(s) = v_0$. The case is similar for $x_0 < 0$.

Therefore the integral curve $\gamma(t) = \beta(t - s)$ solves the same initial value problem as q so by existence and uniqueness

$$q(t) = \gamma(t) = \beta(t - s), \quad \forall\, t \geq 0.$$

Since β, by theorem 6.5, is periodic, so is q □

The above result could also be obtained in a more intuitive manner, simply by realizing that q and β, having the same energy, must lie on the same energy curve. Since the energy curves are the parametrised curves (x,x'), the only thing that can differ is the time.

6.3.1 A Formula for the Period

Determining a theoretical expression of the period T is not difficult given the energy function.

$$\tilde{E}(q,\dot{q}) = \frac{\dot{q}^2}{2} + V(q).$$

Isolate the velocity to obtain

$$\frac{dq}{dt} = \pm\sqrt{2(E - V(q(t)))}.$$

Assume $dq/dt \geq 0$. Then we have a first-order separable differential equation that can be solved using the technique described in [Betounes, 2001, Section 4.2].

The requirement on the velocity directly tells us that this is valid exactly for a quarter of a period; corresponding to the time it takes q to reach its maximal value.

Setting $\psi = \max q(t)$, separating and integrating gives us

$$\int_0^\psi \frac{dq}{\sqrt{2(E - V(q))}} = \int_0^{T/4} dt = \frac{T}{4}.$$

Isolating T we arrive at

$$T = 2\sqrt{2}\int_0^{\psi} \frac{dq}{\sqrt{E - V(q)}} = 2\sqrt{2}\int_0^{\psi} \frac{dq}{\sqrt{V(\psi) - V(q)}} \tag{6.11}$$

Observe that we in theory can derive the period from (6.11) given any initial velocity for which the mechanical energy is negative. However, ψ is of course a function of the initial velocity and the general form of the integral makes it difficult, if not impossible to derive exact values of T given some initial velocity.

CHAPTER 7

Concluding Remarks — Generalization

Although some of our results on the dynamics have explicitly used the specific form of our system, most of these are based on a few simple observations that apply to a much wider range of conservative systems.

Given the general conservative system

$$\frac{d^2x}{dt^2}(t) = F(x(t)) = -V'(x(t)), \quad t \geq 0,$$

where $V : \mathbb{R} \to \mathbb{R}$ is a C^1-function, the following properties are sufficient to obtain the same system dynamics as for our problem

(i) $V'(x)$ is bounded on $\mathbb{R}$ and locally Lipschitz continuous.

(ii) $V(x) < 0$ in some interval I such that $V(x) = 0$ when $x \geq \sup I$ and $x \leq \inf I$. Since V is continuous, by [Apostol, 1974, Theorem 5.10], there exists a minimum value φ such that $V'(\varphi) = 0$. By translation, we can assume $\varphi = 0$.

(iii) V is strictly increasing in the interval $]\varphi; \sup I[$ and monontically decreasing in $]\inf I; \varphi[$.

Existence of unique solutions for (1.3) on p. 1 for all $t \geq 0$ and every initial condition is a consequence of F being bounded and locally Lipschitz continuous. Again we can also find two unstable fixed points. This can be shown exactly as we have done, since our proof only relied on the fact that an explicit formula for the flow was easily obtainable in the regions where $F(x) = 0$.

Regarding the origin this will be a stable fixed point. Here our method does not necessarily apply in general, since we have made some use of the specific representation of the system. In some cases, a linearization technique will be adequate (cf. [Betounes, 2001, Section 6.3 and 7.2-7.3]), so that the stability problem reduces to the problem of determining stability for a linear system. Otherwise, one could attempt to determine a *Liapunov function* for the system, cf. [Betounes, 2001, Section 7.4]. Periodicity follows from our own proof which only used the properties of V listed above.

A simple, but certainly non-trivial, example is the system with V given by

$$V(x) := \begin{cases} 0 & \text{if } x \leq -\pi \text{ or } x \geq \pi; \\ -1 - \cos x & \text{if } -\pi < x < \pi. \end{cases} \tag{7.1}$$

and F correspondingly

$$F(x) := -V'(x) = \begin{cases} 0 & \text{if } x \leq -\pi \text{ or } x \geq \pi; \\ -\sin x & \text{if } -\pi < x < \pi, \end{cases}$$

This is in fact the differential equation describing a variant of the nonlinear pendulum in a restricted region, cf. [Betounes, 2001, p. 8]. It can readily be verified that V fulfils the demands listed on the previous page; and therefore the dynamics are very alike to those of our own system. This can be inspected graphically from a plot of some of the integral curves of the corresponding first order system — this plot is obtained exactly as we did in section 1.1.2. The fact that it is possible to generalise the results obtained to a wide range of seemingly completely different systems indeed shows the power of doing a qualitative analysis of the system dynamics, rather than a quantitive.

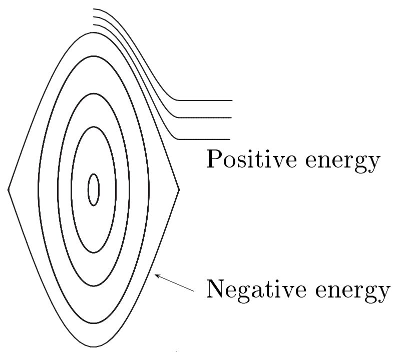

Figure 7.1: *A plot of a few integral curves (in the position-velocity plane) for the system with V given by* (7.1)

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